The 26th power of Dedekind's η-function

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1 Introduction

Dedekind's η-function is defined by

$$\eta(z) = q^{\frac{1}{24}} \prod_{j=1}^{\infty}(1-q^j), \quad \text{where } q = e^{2\pi iz}, \quad \text{Im}(z) > 0.$$

For certain values of d, the expansion of $\eta^d(z)$ in powers of q has a particularly simple form. For example, the cases $d = 1$ and $d = 3$ are due to L. Euler and C. G. J. Jacobi, respectively:

$$\begin{aligned} \eta(24z) &= \sum_{j=-\infty}^{\infty} (-1)^j q^{(6j+1)^2}, \\ \eta^3(8z) &= \sum_{j=-\infty}^{\infty} (4j+1) q^{(4j+1)^2}. \end{aligned}$$

For $d = 8$ we have

$$\begin{aligned} \eta^8(12z) &= \frac{1}{2} \sum_{i=-\infty}^{\infty} (2i)^2 q^{3(2i)^2} \sum_{j=-\infty}^{\infty} (6j-2) q^{(6j-2)^2} \\ &\quad + \frac{1}{2} \sum_{i=-\infty}^{\infty} (2i+1)^2 q^{3(2i+1)^2} \sum_{j=-\infty}^{\infty} (6j+1) q^{(6j+1)^2}, \end{aligned}$$

and there are analogous formulas for $d = 10$ and 14 in terms of double sums. The case $d = 8$ was first considered by F. Klein and R. Fricke [7, p. 373]. The formula for $d = 10$ was discovered by L. Winquist [17], who used it to give an elementary proof of S. Ramanujan's congruence

$$p(11m + 6) \equiv 0 \,(\text{mod } 11),$$

where $p(n)$ is the number of partitions of n. F. J. Dyson [5, p. 637] reports that the case $d = 14$ was discovered by A. O. L. Atkin, and furthermore that Atkin had a formula for $d = 26$. Apart from a special case cited by Dyson [5, p. 651], no details of Atkin's work on $d = 26$ have been published.

Dyson found that there are elegant multiple series expansions for $d = 3$, 8, 10, 14, 15, 21, 24, 26, 28, 35, 36, ... At about the same time, I. G. Macdonald [10] discovered there is an elegant multiple series expansion for any value of d which is the dimension of a finite dimensional simple Lie algebra. Macdonald's results include all of the numbers on Dyson's list, except $d = 26$. In [8] and [9], V. E. Leininger and S. C. Milne utilized [11] and multiple basic hypergeometric series techniques to derive new non-trivial explicit multiple series expansions for additional infinite families of values of d not in [10], but not for $d = 26$. They

also simplified Macdonald's results corresponding to affine root systems of type A_ℓ.

The purpose of this article is to prove a formula for $\eta^{26}(z)$ in terms of a double series. A special case of our formula is as follows. Suppose $12n+13$ is prime. Let $a+ib$ and $c+id$ be the unique Gaussian integers which satisfy the conditions

$$a^2+b^2 = 24n+26, \quad a,\, b \equiv 1 \,(\mathrm{mod}\ 6), \quad a<b,$$
$$c^2+3d^2 = 36n+39, \quad d \equiv 1 \,(\mathrm{mod}\ 6), \quad c>0.$$

Then the coefficient of q^n in $\prod_{j=1}^{\infty}(1-q^j)^{26}$ is

$$\frac{1}{2^6 3^4 11^2 13}\left(\frac{(-1)^{(a+b-2)/6}}{2^6}\,\mathrm{Re}\left((a+ib)^{12}\right)+\frac{(-1)^{(c+d-1)/6}}{3^6}\,\mathrm{Re}\left((c+id\sqrt{3})^{12}\right)\right).$$

This is different from the formula of Atkin quoted by Dyson. As an example, when $n=2$ we have $a+ib=-5+7i$ and $c+id=6-5i$, so the coefficient of q^2 in $\prod_{j=1}^{\infty}(1-q^j)^{26}$ is therefore

$$\begin{aligned}&\frac{1}{2^6 3^4 11^2 13}\left(\frac{1}{64}\,\mathrm{Re}\left((-5+7i)^{12}\right)+\frac{1}{729}\,\mathrm{Re}\left((6-5i\sqrt{3})^{12}\right)\right)\\ &=\frac{1}{2^6 3^4 11^2 13}(1025046359+1413128809)\\ &=299.\end{aligned}$$

Our proof is based on the observation that $\eta^2(z)$ may be expressed as a product of two theta functions in two different ways:

$$\eta^2(z)=\left(\sum_{j=-\infty}^{\infty}(-1)^j q^{(6j+1)^2/24}\right)^2=\left(\sum_{j=-\infty}^{\infty}(-1)^j q^{(6j+1)^2/12}\right)\left(\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2}\right).$$

Atkin's proof [1] uses properties of $\eta^{10}(z)E_4^2(z)$ and $\eta^{14}(z)E_6(z)$, where E_4 and E_6 are Eisenstein series of weights 4 and 6, respectively. Atkin's notes [1] indicate that he discovered his formula for $\eta^{26}(z)$ in 1965, and in 1966 he found another formula, different from the one quoted by Dyson [5, p. 651]. For a published proof of a formula for $\eta^{26}(z)$, see the paper by J.-P. Serre [16].

2 Statement of results

Let m and n be real numbers and define

$$\begin{aligned}f(m,n) &= \sum_{j=0}^{6}\binom{12}{2j}(-1)^j m^j n^{6-j}\\ &= m^6-66m^5n+495m^4n^2-924m^3n^3+495m^2n^4-66mn^5+n^6.\end{aligned}$$

Observe that

$$f(m^2, n^2) = \mathrm{Re}\left((m+in)^{12}\right).$$

Ramanujan's Eisenstein series are

$$\begin{aligned} P &= 1 - 24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}, \\ Q &= 1 + 240\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j}, \\ R &= 1 - 504\sum_{j=1}^{\infty}\frac{j^5q^j}{1-q^j}. \end{aligned}$$

We will prove the following identities:

Lemma 1

$$\begin{aligned} &\eta^2(z)(3999Q^3 - 4000R^2) \\ &\quad = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j} f\left(\frac{(6i+1)^2}{2}, \frac{(6j+1)^2}{2}\right) q^{[(6i+1)^2+(6j+1)^2]/24}. \end{aligned}$$

Lemma 2

$$\begin{aligned} &\eta^2(z)(5439Q^3 - 5438R^2) \\ &\quad = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j} f\left(12i^2, (6j+1)^2\right) q^{i^2+(6j+1)^2/12}. \end{aligned}$$

If we add these results and use the fact that [13], [15, pp. 140 – 144]

$$Q^3 - R^2 = 1728\eta^{24}(z),$$

we obtain our main result:

Theorem 3

$$\begin{aligned} &16308864\,\eta^{26}(z) \\ &= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j} f\left(\frac{(6i+1)^2}{2}, \frac{(6j+1)^2}{2}\right) q^{[(6i+1)^2+(6j+1)^2]/24} \\ &\quad + \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j} f\left(12i^2, (6j+1)^2\right) q^{i^2+(6j+1)^2/12}. \end{aligned}$$

By comparing coefficients on both sides we readily obtain:

Corollary 4
Let

$$\eta^r(z) = q^{r/24} \sum_{n=0}^{\infty} p_r(n) q^n,$$

where the coefficients $p_r(n)$ are defined by

$$\prod_{m=1}^{\infty} (1-q^m)^r = \sum_{n=0}^{\infty} p_r(n) q^n.$$

Then

$$\begin{aligned} 16308864\, p_{26}(n) &= \sum_{\substack{\alpha^2+\beta^2=24n+26 \\ \alpha,\, \beta \equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-2)/6} f\left(\frac{\alpha^2}{2}, \frac{\beta^2}{2}\right) \\ &\quad + \sum_{\substack{\gamma^2+3\delta^2=36n+39 \\ \gamma \equiv 0 \pmod 6,\, \delta \equiv 1 \pmod 6}} (-1)^{(\gamma+\delta-1)/6} f\left(\frac{\gamma^2}{3}, \delta^2\right). \end{aligned}$$

3 Proof of Lemma 1

Let

$$V_{2\ell} = \frac{\displaystyle\sum_{j=-\infty}^{\infty} (-1)^j (6j+1)^{2\ell} q^{(6j+1)^2/24}}{\displaystyle\sum_{j=-\infty}^{\infty} (-1)^j q^{(6j+1)^2/24}}.$$

Using the relation

$$V_{2\ell+2} = PV_{2\ell} + 24q \frac{dV_{2\ell}}{dq}$$

and the Ramanujan differential equations for P, Q and R, Ramanujan [14, p. 369] showed that

$$\begin{aligned} V_0 &= 1, \\ V_2 &= P, \\ V_4 &= 3P^2 - 2Q, \\ V_6 &= 15P^3 - 30PQ + 16R, \\ V_8 &= 105P^4 - 420P^2Q + 448PR - 132Q^2, \end{aligned}$$

$$
\begin{aligned}
V_{10} &= 945P^5 - 6300P^3Q + 10080P^2R - 5940PQ^2 + 1216QR, \\
V_{12} &= 10395P^6 - 103950P^4Q + 221760P^3R \\
&\quad -196020P^2Q^2 + 80256PQR - 2712Q^3 - 9728R^2. \qquad (1)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&V_0V_{12} - 66V_2V_{10} + 495V_4V_8 - 924V_6^2 + 495V_8V_4 - 66V_{10}V_2 + V_{12}V_0 \\
&\qquad = 64(3999Q^3 - 4000R^2).
\end{aligned}
$$

If we multiply this by $\eta^2(z)/64$ we complete the proof of Lemma 1.

□

4 Proof of Lemma 2

The key to proving Lemma 2 is to write

$$
\begin{aligned}
\eta^2(z) &= q^{\frac{1}{12}} \prod_{j=1}^{\infty}(1-q^j)^2 \\
&= q^{\frac{1}{12}} \prod_{j=1}^{\infty}(1-q^{2j}) \prod_{j=1}^{\infty} \frac{(1-q^j)^2}{(1-q^{2j})} \\
&= \left(\sum_{j=-\infty}^{\infty} (-1)^j q^{(6j+1)^2/12} \right) \left(\sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} \right). \qquad (2)
\end{aligned}
$$

Let

$$
\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}, \qquad \psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2},
$$

and define

$$
z = \varphi(q)^2, \qquad x = 16q\frac{\psi(q^2)^4}{\varphi(q)^4}.
$$

Let

$$
\begin{aligned}
\overline{V}_{2\ell} &= \frac{\displaystyle\sum_{j=-\infty}^{\infty} (-1)^j (6j+1)^{2\ell} q^{(6j+1)^2/12}}{\displaystyle\sum_{j=-\infty}^{\infty} (-1)^j q^{(6j+1)^2/12}} = V_{2\ell}(q^2), \\
W_{2\ell} &= \frac{\displaystyle\sum_{j=-\infty}^{\infty} (-1)^j (12j^2)^{\ell} q^{j^2}}{\displaystyle\sum_{j=-\infty}^{\infty} (-1)^j q^{j^2}}.
\end{aligned}
$$

We will express $\overline{V}_{2\ell}$ and $W_{2\ell}$ in terms of P, z and x.

From [3, pp. 126–127] we have

$$\begin{aligned} P(q^2) &= \frac{1}{2}\left(P + z^2(1+x)\right), \\ Q(q^2) &= z^4(1 - x + x^2), \\ R(q^2) &= z^6(1+x)(1-\frac{x}{2})(1-2x). \end{aligned}$$

Using these in (1) we obtain

$$\begin{aligned} \overline{V}_0 &= 1, \\ 2\overline{V}_2 &= P + z^2(1+x), \\ 4\overline{V}_4 &= 3P^2 + 6Pz^2(1+x) - z^4(5 - 14x + 5x^2), \\ 8\overline{V}_6 &= 15P^3 + 45P^2z^2(1+x) - 15z^4P(5 - 14x + 5x^2) \\ &\quad + z^6(1+x)(23x^2 - 170x + 23), \\ 16\overline{V}_8 &= 105P^4 + 420P^3z^2(1+x) - 210P^2z^4(5 - 14x + 5x^2) \\ &\quad + 28Pz^6(1+x)(23 - 170x + 23x^2) \\ &\quad - z^8(103 - 1172x + 16458x^2 - 1172x^3 + 103x^4), \\ 32\overline{V}_{10} &= 945P^5 + 4725P^4z^2(1+x) - 3150P^3z^4(5 - 14x + 5x^2) \\ &\quad + 630P^2z^6(1+x)(23 - 170x + 23x^2) \\ &\quad - 45Pz^8(103 - 1172x + 16458x^2 - 1172x^3 + 103x^4) \\ &\quad + z^{10}(1+x)(257 - 7852x - 346266x^2 - 7852x^3 + 257x^4), \\ 64\overline{V}_{12} &= 10395P^6 + 62370P^5z^2(1+x) - 51975P^4z^4(5 - 14x + 5x^2) \\ &\quad + 13860P^3z^6(1+x)(23 - 170x + 23x^2) \\ &\quad - 1485P^2z^8(103 - 1172x + 16458x^2 - 1172x^3 + 103x^4) \\ &\quad + 66Pz^{10}(1+x)(257 - 7852x - 346266x^2 - 7852x^3 + 257x^4) \\ &\quad + z^{12}(4387 + 12282x - 10840467x^2 - 17010388x^3 - 10840467x^4 \\ &\qquad + 12282x^5 + 4387x^6). \end{aligned}$$

Now we will express $W_{2\ell}$ in terms of P, z and x. First, observe that $W_0 = 1$. Next, from [3, pp. 120–129] we have

$$\varphi(-q) = z^{\frac{1}{2}}(1-x)^{\frac{1}{4}},$$

$$q\frac{dx}{dq} = z^2x(1-x),$$

$$12q\frac{dz}{dq} = Pz + z^3(5x-1),$$

$$Q = z^4(1+14x+x^2),$$

$$R = z^6(1+x)(1-34x+x^2).$$

Using these we obtain

$$\begin{aligned} W_2 &= 12q\frac{d}{dq}\log\sqrt{\varphi(-q)} \\ &= 3q\frac{d}{dq}\log z(1-x)^{\frac{1}{2}} \\ &= \frac{3}{z}\,q\frac{dz}{dq} - \frac{3}{2(1-x)}\,q\frac{dx}{dq} \\ &= \frac{1}{2}\left(P - z^2 - z^2x\right). \end{aligned}$$

If we apply $q\frac{d}{dq}$ to the equation defining $W_{2\ell}$ and simplify, we obtain the differential recurrence relation

$$W_{2\ell+2} = W_2W_{2\ell} + 12q\frac{d}{dq}W_{2\ell}.$$

Additional values of $W_{2\ell}$ can be computed using the differential recurrence relation together with Ramanujan's differential equations for P, Q and R. We obtain:

$$\begin{aligned} \frac{4}{3}W_4 &= P^2 - 2Pz^2(1+x) + z^4(1-22x+x^2), \\ \frac{8}{3}W_6 &= 5P^3 - 15P^2z^2(1+x) + 15Pz^4(1-22x+x^2) \\ &\quad -z^6(1+x)(5+226x+5x^2), \\ \frac{16}{3}W_8 &= 35P^4 - 140P^3z^2(1+x) + 210P^2z^4(1-22x+x^2) \\ &\quad -28Pz^6(1+x)(5+226x+5x^2) \\ &\quad +z^8(35-2596x-6990x^2-2596x^3+35x^4), \\ \frac{32}{27}W_{10} &= 35P^5 - 175P^4z^2(1+x) + 350P^3z^4(1-22x+x^2) \\ &\quad -70P^2z^6(1+x)(5+226x+5x^2) \\ &\quad +5Pz^8(35-2596x-6990x^2-2596x^3+35x^4) \end{aligned}$$

$$-z^{10}(1+x)(35-196x+18546x^2-196x^3+35x^4),$$

$$\begin{aligned}\frac{64}{27}W_{12} &= 385P^6-2310P^5z^2(1+x)+5775P^4z^4(1-22x+x^2)\\ &\quad -1540P^3z^6(1+x)(5+226x+5x^2)\\ &\quad +165P^2z^8(35-2596x-6990x^2-2596x^3+35x^4)\\ &\quad -66Pz^{10}(1+x)(35-196x^3+18546x^2-196x^3+35x^4)\\ &\quad +z^{12}(385+18078x-50385x^2-841180x^3-50385x^4+18078x^5+385x^6).\end{aligned}$$

Observe that

$$\begin{aligned}&\overline{V}_0W_{12}-66\overline{V}_2W_{10}+495\overline{V}_4W_8-924\overline{V}_6W_6+495\overline{V}_8W_4-66\overline{V}_{10}W_2+\overline{V}_{12}W_0\\ =\;& z^{12}(1+587346x-2348625x^2+3526652x^3-2348625x^4+587346x^5+x^6)\\ =\;& 5439z^{12}(1+14x+x^2)^3-5438z^{12}(1+x)^2(1-34x+x^2)^2\\ =\;& 5439Q^3-5438R^2.\end{aligned}$$

If we multiply this by $\eta^2(z)$ and use (2) we complete the proof of Lemma 2.

□

5 Consequences

5.1 Lacunarity

A series $q^\nu\sum_{n=0}^{\infty}a_nq^n$ is called lacunary if almost all of the coefficients a_n are zero, i.e., $\lim_{N\to\infty}\frac{M(N)}{N}=0$, where $M(N)$ is the number of $n\le N$ with $a_n\neq 0$. Serre [16] showed that the only even values of d for which $\eta^d(z)$ is lacunary are $d=2,\ 4,\ 6,\ 8,\ 10,\ 14$ or 26. It is still unknown if there are any odd values of d, besides $d=1$ and 3, for which $\eta^d(z)$ is lacunary.

The series representation given in Theorem 3, together with a theorem of Landau [2, p. 244, Th. 10.5], imply that $\eta^{26}(z)$ is lacunary.

5.2 The Hecke operator

If $p\equiv 11\pmod{12}$, then Corollary 4 together with the elementary method used in [4] imply

$$p_{26}\left(pn+\frac{13}{12}(p^2-1)\right)=p^{12}p_{26}\left(\frac{n}{p}\right).$$

This provides an elementary derivation of the case $r=26$ of the following theorem of Newman [12]:

Theorem 5

Suppose that r is one of the numbers 2, 4, 6, 8, 10, 14, 26. Let p be a prime > 3 such that $r(p+1) \equiv 0 \pmod{24}$. Let $\Delta = (p^2-1)/24$, and define $p_r(x) = 0$ if x is not a non-negative integer. Then

$$p_r(pn + r\Delta) = (-p)^{(r/2)-1} p_r\left(\frac{n}{p}\right).$$

Furthermore there are no other values of r for which the theorem is true.

Elementary proofs for the cases $r = 2$, 4, 6, 8, 10 and 14 of Newman's theorem were given in [4].

5.3 Values of n for which $p_{26}(n) = 0$

The explicit formula in Corollary 4 may be used to prove two conditions given by Serre [16, p. 213] which imply $p_{26}(n) = 0$.

Proposition 6
Suppose that the factorization of $12n+13$ into distinct primes contains at least one prime congruent to -1 modulo 4 raised to an odd power, and also at least one prime congruent to -1 modulo 3 raised to an odd power. Then $p_{26}(n) = 0$.

Proof
Let $12n+13 = \prod_p p^{\lambda_p}$ be the factorization of $12n+13$ into primes. Let $r_2(m)$ and $s(m)$ denote the number of solutions in integers of $x^2+y^2 = m$ and $x^2+3y^2 = m$, respectively. Two classical results [6] are, for any positive integer m,

$$r_2(m) = 4\left(d_{1,4}(m) - d_{3,4}(m)\right), \tag{3}$$

$$s(m) = 2(d_{1,3}(m) - d_{2,3}(m)) + 4(d_{4,12}(m) - d_{8,12}(m)), \tag{4}$$

where $d_{j,k}(m)$ denotes the number of divisors of m which are congruent to j modulo k. It follows that

$$r_2(24n+26) = 4 \prod_{p \equiv 1 \pmod 4} (\lambda_p + 1) \prod_{p \equiv -1 \pmod 4} \frac{1+(-1)^{\lambda_p}}{2}, \tag{5}$$

$$s(36n+39) = 2 \prod_{p \equiv 1 \pmod 3} (\lambda_p + 1) \prod_{p \equiv -1 \pmod 3} \frac{1+(-1)^{\lambda_p}}{2}. \tag{6}$$

Consequently, $r_2(24n+26) = s(36n+39) = 0$. Therefore the sums in Corollary 4 are empty and it follows that $p_{26}(n) = 0$.

□

Proposition 7

Suppose $12n+13$ *is a square and all the prime factors of* $12n+13$ *are congruent to* -1 *modulo* 12. *Then* $p_{26}(n) = 0$.

Proof
Write $12n + 13 = x^2 = \prod_p p^{\lambda_p}$, where the product is over primes $p \equiv -1 \pmod{12}$, and all the exponents λ_p are even. Without loss of generality we can assume $x \equiv 1 \pmod{12}$. Then (5) implies $r_2(26n+24) = 4$, and in fact the representations are $24n+26 = (\pm x)^2 + (\pm x)^2$. Similarly, (6) implies $s(36n+39) = 2$, and the representations are $36n + 39 = (0)^2 + 3(\pm x)^2$. Corollary 4 implies

$$16308864\, p_{26}(n) = f\left(\frac{x^2}{2}, \frac{x^2}{2}\right) + f(0, x^2) = x^{12} - x^{12} = 0.$$

□

5.4 The case when $12n + 13$ is prime

For this section, let us consider the case when $12n + 13$ is prime, in which case we write $p = 12n + 13$. Corollary 4 implies

$$\begin{aligned} 16308864\, p_{26}(n) \quad = \quad & \sum_{\substack{\alpha^2+\beta^2=2p \\ \alpha,\, \beta \equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-2)/6} f\left(\frac{\alpha^2}{2}, \frac{\beta^2}{2}\right) \\ & + \sum_{\substack{\gamma^2+3\delta^2=3p \\ \gamma \equiv 0 \pmod 6,\, \delta \equiv 1 \pmod 6}} (-1)^{(\gamma+\delta-1)/6} f\left(\frac{\gamma^2}{3}, \delta^2\right), \end{aligned}$$

Equation (3) implies that $\alpha^2 + \beta^2 = 2p$ has exactly eight solutions in integers, say $(\alpha, \beta) = (\pm a, \pm b)$, $(\pm b, \pm a)$, where $a,\, b \equiv 1 \pmod 6$ and $a < b$. Two of these eight solutions satisfy the conditions $\alpha,\, \beta \equiv 1 \pmod 6$, namely $(\alpha, \beta) = (a, b)$ and $(\alpha, \beta) = (b, a)$. Therefore

$$\sum_{\substack{\alpha^2+\beta^2=2p \\ \alpha,\beta \equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-2)/6} f\left(\frac{\alpha^2}{2}, \frac{\beta^2}{2}\right) = 2(-1)^{(a+b-2)/6} f\left(\frac{a^2}{2}, \frac{b^2}{2}\right).$$

Similarly, equation (4) implies that $\gamma^2 + 3\delta^2 = 3p$ has exactly four solutions in integers, say $(\gamma, \delta) = (\pm c, \pm d)$, where $c > 0$ and $d \equiv 1 \pmod 6$. Two of these four solutions satisfy $\gamma \equiv 0 \pmod 6$, $\delta \equiv 1 \pmod 6$, namely $(\gamma, \delta) = (c, d)$ and $(\gamma, \delta) = (-c, d)$. Therefore

$$\sum_{\substack{\gamma^2+3\delta^2=3p \\ \gamma \equiv 0 \pmod 6,\, \delta \equiv 1 \pmod 6}} (-1)^{(\gamma+\delta-1)/6} f\left(\frac{\gamma^2}{3}, \delta^2\right) = 2(-1)^{(c+d-1)/6} f\left(\frac{c^2}{3}, d^2\right).$$

Therefore we have proved a result which is equivalent to the formula mentioned in the introduction:

Theorem 8
Let $p = 12n + 13$ be prime. Let (a, b) be the unique solution in integers of

$$a^2 + b^2 = 2p, \quad a,\, b \equiv 1 \,(\mathrm{mod}\ 6), \quad a < b,$$

and let (c, d) be the unique solution in integers of

$$c^2 + 3d^2 = 3p, \quad d \equiv 1 \,(\mathrm{mod}\ 6), \quad c > 0.$$

Then

$$8154432\, p_{26}(n) = (-1)^{(a+b-2)/6} f\left(\frac{a^2}{2}, \frac{b^2}{2}\right) + (-1)^{(c+d-1)/6} f\left(\frac{c^2}{3}, d^2\right). \quad (7)$$

□

5.5 Remark

We note that

$$f(m^2, n^2) = \prod_{j=1}^{6} (m^2 - a_j^2 n^2)$$

where

$$\begin{array}{ll} a_1 = \sqrt{2} + 1, & a_2 = \sqrt{2} - 1, \\ a_3 = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2, & a_4 = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2, \\ a_5 = \sqrt{6} - \sqrt{3} - \sqrt{2} + 2, & a_6 = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2, \end{array}$$

and

$$a_1 a_2 = a_3 a_4 = a_5 a_6 = 1.$$

As a result, (7) may be written as a sum of two expressions, each of which factors into linear factors involving a and b, and c and d, respectively.

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