DIFFERENTIAL EQUATIONS SATISFIED BY EISENSTEIN SERIES OF LEVEL 2

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Abstract. Ramanujan's differential equations for the classical Eisenstein series are of great importance to many areas in number theory and special functions. H.H. Chan recently demonstrated that these differential equations can be derived from the triple product identity and the quintuple product identity in an elementary manner. In this article, we extend this method in a uniform manner to derive corresponding differential equations for the Eisenstein series of level 2. Several applications of these differential equations are also given.

1. Introduction

For $|x| < 1$ and non-negative integers r and s, S. Ramanujan [23] defined

$$\Phi_{r,s} := \sum_{m,n=1}^{\infty} m^r n^s x^{mn}, \tag{1.1}$$

and

$$P := P(x) = 1 - 24\Phi_{0,1} = 1 - 24\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}, \tag{1.2a}$$

$$Q := Q(x) = 1 + 240\Phi_{0,3} = 1 + 240\sum_{n=1}^{\infty} \frac{n^3x^n}{1-x^n}, \tag{1.2b}$$

$$R := R(x) = 1 - 504\Phi_{0,5} = 1 - 504\sum_{n=1}^{\infty} \frac{n^5x^n}{1-x^n}. \tag{1.2c}$$

P, Q and R are the classical (normalized) Eisenstein series of weight 2, 4 and 6 respectively. Using the following pair of trigonometric identities

$$\left(\frac{1}{4}\cot\frac{u}{2} + \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}\sin nu\right)^2$$
$$= \left(\frac{1}{4}\cot\frac{u}{2}\right)^2 + \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)^2}\cos nu + \frac{1}{2}\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n}(1-\cos nu) \tag{1.3a}$$

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and

$$\left(\frac{1}{8}\cot^2\frac{u}{2}+\frac{1}{12}\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}(1-\cos nu)\right)^2$$
$$=\left(\frac{1}{8}\cot^2\frac{u}{2}+\frac{1}{12}\right)^2+\frac{1}{12}\sum_{n=1}^{\infty}\frac{n^3x^n}{1-x^n}(5+\cos nu), \tag{1.3b}$$

Ramanujan derived many relations between $\Phi_{r,s}$ and P, Q and R. In particular, he proved the following differential equations

$$x\frac{\mathrm{d}P}{\mathrm{d}x}=\frac{P^2-Q}{12}, \tag{1.4a}$$
$$x\frac{\mathrm{d}Q}{\mathrm{d}x}=\frac{PQ-R}{3}, \tag{1.4b}$$
$$x\frac{\mathrm{d}R}{\mathrm{d}x}=\frac{PR-Q^2}{2}. \tag{1.4c}$$

Remark. The left hand side of (1.3a) is essentially the square of the logarithmic derivative of the Jacobi theta function θ_1 (see Section 2 and equation (2.6a)). This identity was later generalized by K. Venkatachaliengar [27, p. 1-13] to one involving two variables. The second trigonometric identity (1.3b) is equivalent to a differential equation satisfied by the Weierstrass $\wp$-function [28, p. 450]. Finally, an exposition of Ramanujan's method can be found in [2, Chpt. 4].

V. Ramamani [21, p. 82-123],[22] extended Ramanujan's method by defining

$$\Psi_{r,s}:=\sum_{m,n=1}^{\infty}(-1)^{n-1}m^rn^sx^{mn}, \tag{1.5}$$

$$F_{r,s}:=\sum_{m,n=1}^{\infty}(2m-1)^rn^sx^{(2m-1)n/2} \tag{1.6}$$

and proving a third trigonometric identity

$$\left(\frac{1}{4}\cot\frac{u}{2}+\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}\sin nu\right)^3$$
$$=\left(\frac{1}{4}\cot\frac{u}{2}\right)^3-\frac{3}{2}\sum_{n=1}^{\infty}\frac{x^n}{(1-x^n)^3}\sin nu+\frac{3}{4}\sum_{n=1}^{\infty}\frac{(n+1)x^n}{(1-x^n)^2}\sin nu$$
$$-\frac{1}{16}\sum_{n=1}^{\infty}\frac{(2n^2+1)x^n}{1-x^n}\sin nu+\frac{3}{8}\cot\frac{u}{2}\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}$$
$$+\frac{3}{2}\left(\sum_{n=1}^{\infty}\frac{x^n}{1-x^n}\sin nu\right)\left(\sum_{n=1}^{\infty}\frac{nx^n}{1-x^n}\right). \tag{1.7}$$

She (see also [13]) used these to prove two families of differential equations involving Eisenstein series, one of which is the following

$$x\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}x} = \frac{\mathcal{E}\mathcal{P} - \mathcal{Q}}{2}, \tag{1.8a}$$

$$x\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x} = \frac{\mathcal{P}^2 - \mathcal{Q}}{4}, \tag{1.8b}$$

$$x\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}x} = \mathcal{P}\mathcal{Q} - \mathcal{E}\mathcal{Q}, \tag{1.8c}$$

where

$$\mathcal{E} := \mathcal{E}(x) = 1 + 24\Psi_{1,0} = 1 + 24\sum_{n=1}^{\infty}\frac{nx^n}{1+x^n}, \tag{1.9a}$$

$$\mathcal{P} := \mathcal{P}(x) = 1 + 8\Psi_{0,1} = 1 + 8\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nx^n}{1-x^n}, \tag{1.9b}$$

$$\mathcal{Q} := \mathcal{Q}(x) = 1 - 16\Psi_{0,3} = 1 - 16\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^3x^n}{1-x^n}. \tag{1.9c}$$

Subsequently, T. Huber [15, Th. 4.2.1] found a fourth family of differential equations using similar methods. In hindsight, we know that these four families are collectively the Eisenstein series of level 2 [17, p. 131]. If we let $q = \mathrm{e}^{2\pi i\tau}$ for some variable τ in the complex upper half plane, then $\mathcal{E}$ is a weight 2 modular form on $\Gamma_0(2)$. (See [3, Lemma 3.3] for a proof.) $\mathcal{P}$ and $\mathcal{Q}$ are the Eisenstein series of weight 2 and 4, associated with $\Gamma_0(2)$, although $\mathcal{P}$ is quasi-modular. H. Hahn [13] and R.S. Maier [20] have given proofs of equations (1.8) using the theory of modular forms.

Recently, H.H. Chan [5] showed that Ramanujan's original differential equations (1.4) can be derived from series expansions of $\theta_1(z)$ and $\theta_1(2z)/\theta_1(z)$. The two identities that Chan used are direct consequences of the Jacobi triple product identity [2, p. 10] and the quintuple product identity [2, p. 18]. Working in terms of Jacobi theta functions has the distinct advantage that there are four natural theta functions! Indeed, the four theta functions correspond exactly to the four families of Eisenstein series of level 2. (See (2.7).) In this article, we will extend Chan's method to give a uniform proof of the results of Ramamani and Huber.

For $|q| < 1$ and $k > 0$, define

$$E_{1,2k} := E_{1,2k}(q) = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^{2n}}{1-q^{2n}}, \tag{1.10a}$$

$$E_{2,2k} := E_{2,2k}(q) = 1 - \frac{4k}{B_{2k}(2^{2k}-1)}\sum_{n=1}^{\infty}\frac{(-1)^n n^{2k-1}q^{2n}}{1-q^{2n}}, \tag{1.10b}$$

$$E_{3,2k} := E_{3,2k}(q) = -\frac{4k}{B_{2k}(2^{2k}-1)}\sum_{n=1}^{\infty}\frac{(-1)^n n^{2k-1}q^{n}}{1-q^{2n}}, \tag{1.10c}$$

$$E_{4,2k} := E_{4,2k}(q) = -\frac{4k}{B_{2k}(2^{2k}-1)}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^{n}}{1-q^{2n}}, \tag{1.10d}$$

where B_k is the Bernoulli number defined by the generating function

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{1.11}$$

These are the four families of Eisenstein series of level 2 with weight $2k$. (They correspond to $G_{2k}^{(0,0)}$, $G_{2k}^{(0,1)}$, $G_{2k}^{(1,1)}$ and $G_{2k}^{(1,0)}$ in the notation of [17, p. 131].) Note that in order to avoid fractional powers, we have followed the classical notation and replaced x by q^2. Thus, in our definition,

$$E_{1,2}(q) = P(x), \qquad E_{1,4}(q) = Q(x), \qquad E_{1,6}(q) = R(x),$$
$$E_{2,2}(q) = \mathcal{P}(x), \qquad E_{2,4}(q) = \mathcal{Q}(x).$$

All of the weight 2 Eisenstein series are not modular forms. However, in the case of $j = 2, 3$ and 4, we can define a 'corrected' series as follows,

$$f_j := f_j(q) = \frac{3E_{j,2} - E_{1,2}}{2}. \tag{1.12}$$

We then have $f_2(q) = \mathcal{E}(x)$, after some series rearrangement.

The main results of this article are the following.

Theorem 1.1. *For $j = 2, 3$ or 4, we have*

$$q\frac{\mathrm{d}f_j}{\mathrm{d}q} = f_j E_{j,2} - E_{j,4}, \tag{1.13a}$$

$$q\frac{\mathrm{d}E_{j,2}}{\mathrm{d}q} = \frac{E_{j,2}^2 - E_{j,4}}{2}, \tag{1.13b}$$

$$q\frac{\mathrm{d}E_{j,4}}{\mathrm{d}q} = 2\left(E_{j,2}E_{j,4} - E_{j,6}\right). \tag{1.13c}$$

Theorem 1.2. *For $j = 2, 3$ or 4, we have*

$$E_{j,6} = f_j E_{j,4}. \tag{1.14}$$

Remark. For $j = 2$, Theorem 1.2 implies that equations (1.13) are equivalent to (1.8). The cases $j = 2$ and 4 were proved by Ramamani [21, 22] while the case $j = 3$ was proved by Huber [15, Th. 4.2.1].

2. Proofs of results

In this section, we give the definitions of several families of theta series that are required for the proof of Theorems 1.1 and 1.2. For $|q| < 1$, the Jacobi theta functions [28, p. 464] are

$$\theta_1(z) := \theta_1(z|q^2) = 2\sum_{k=0}^{\infty}(-1)^k q^{(2k+1)^2/4}\sin((2k+1)z), \tag{2.1a}$$

$$\theta_2(z) := \theta_2(z|q^2) = 2\sum_{k=0}^{\infty} q^{(2k+1)^2/4}\cos((2k+1)z), \tag{2.1b}$$

$$\theta_3(z) := \theta_3(z|q^2) = 1 + 2\sum_{k=1}^{\infty} q^{k^2}\cos(2kz), \tag{2.1c}$$

$$\theta_4(z) := \theta_4(z|q^2) = 1 + 2\sum_{k=1}^{\infty}(-1)^k q^{k^2}\cos(2kz). \tag{2.1d}$$

If we expand the first equation as a series in z and interchange the order of summation, we get

$$\theta_1(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(2k-1)!} \left(2 \sum_{n=0}^{\infty} (-1)^n (2n+1)^{2k-1} q^{(2n+1)^2/4} \right). \tag{2.2}$$

This can be done for the other three equations, leading to

$$\theta_1(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(2k-1)!} S_{1,2k-1}\,, \tag{2.3a}$$

$$\theta_j(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k} z^{2k}}{(2k)!} S_{j,2k}\,, \qquad j = 2, 3, 4, \tag{2.3b}$$

where $S_{j,k}$ for non-negative integers j and k are defined by

$$S_{1,2k-1} := 2 \sum_{n=0}^{\infty} (-1)^n (2n+1)^{2k-1} q^{(2n+1)^2/4}, \tag{2.4a}$$

$$S_{2,2k} := 2 \sum_{n=0}^{\infty} (2n+1)^{2k} q^{(2n+1)^2/4}, \tag{2.4b}$$

$$S_{3,0} := 1 + 2 \sum_{n=1}^{\infty} q^{n^2}, \qquad S_{3,2k} := 2 \sum_{n=1}^{\infty} (2n)^{2k} q^{n^2}, \tag{2.4c}$$

$$S_{4,0} := 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}, \qquad S_{4,2k} := 2 \sum_{n=1}^{\infty} (-1)^n (2n)^{2k} q^{n^2}. \tag{2.4d}$$

For these $S_{j,k}$, we observe that

$$4q \frac{\mathrm{d}S_{j,k}}{\mathrm{d}q} = S_{j,k+2}. \tag{2.5}$$

Next, we take the logarithmic derivative of the Jacobi theta functions [28, p. 489].

$$\frac{\theta_1'}{\theta_1}(z) = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin(2nz), \tag{2.6a}$$

$$\frac{\theta_2'}{\theta_2}(z) = -\tan z + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{1-q^{2n}} \sin(2nz), \tag{2.6b}$$

$$\frac{\theta_3'}{\theta_3}(z) = 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n}}{1-q^{2n}} \sin(2nz), \tag{2.6c}$$

$$\frac{\theta_4'}{\theta_4}(z) = 4 \sum_{n=1}^{\infty} \frac{q^{n}}{1-q^{2n}} \sin(2nz), \tag{2.6d}$$

where the notation θ_j' denotes the derivative with respect to z. We again expand these series in terms of z and interchange summation to obtain

$$\frac{\theta_1'}{\theta_1}(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} E_{1,2k} z^{2k-1}, \tag{2.7a}$$

$$\frac{\theta_j'}{\theta_j}(z) = \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k}(2^{2k}-1) B_{2k}}{(2k)!} E_{j,2k} z^{2k-1}, \qquad j = 2, 3, 4. \tag{2.7b}$$

We now define the third order analogues of $S_{j,k}$ by

$$T_{1,2k} := \sum_{n=-\infty}^{\infty} (-1)^n (6n+1)^{2k} q^{(6n+1)^2/12}, \tag{2.8a}$$

$$T_{2,2k-1} := \sum_{n=-\infty}^{\infty} (6n+1)^{2k-1} q^{(6n+1)^2/12}, \tag{2.8b}$$

$$T_{3,2k-1} := \sum_{n=-\infty}^{\infty} (6n+2)^{2k-1} q^{(6n+2)^2/12}, \tag{2.8c}$$

$$T_{4,2k-1} := \sum_{n=-\infty}^{\infty} (-1)^n (6n+2)^{2k-1} q^{(6n+2)^2/12}. \tag{2.8d}$$

One can check that $T_{j,k}$ defined above satisfies

$$12q \frac{\mathrm{d}T_{j,k}}{\mathrm{d}q} = T_{j,k+2}. \tag{2.9}$$

These $T_{j,k}$ arise from the series expansions of a product of three distinct Jacobi theta functions.

Theorem 2.1. *There holds*

$$\theta_2(z)\theta_3(z)\theta_4(z) = K(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/12} \cos((6n+1)z), \tag{2.10a}$$

$$\theta_1(z)\theta_3(z)\theta_4(z) = K(q) \sum_{n=-\infty}^{\infty} q^{(6n+1)^2/12} \sin((6n+1)z), \tag{2.10b}$$

$$\theta_1(z)\theta_2(z)\theta_4(z) = K(q) \sum_{n=-\infty}^{\infty} q^{(6n+2)^2/12} \sin((6n+2)z), \tag{2.10c}$$

$$\theta_1(z)\theta_2(z)\theta_3(z) = K(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+2)^2/12} \sin((6n+2)z), \tag{2.10d}$$

where the normalizing constant

$$K(q) = 2q^{1/6} \prod_{n=1}^{\infty} (1-q^{2n})^2. \tag{2.11}$$

Equations (2.10) are four formulations of the BC_1 Macdonald identity [19], [26, Eq. (3.2), (3.13), (3.14) and (3.6)]. They are equivalent forms of what is commonly known as the quintuple product identity. All four appeared in [25] and (2.10a) was studied in [18]. Using the technique of Carlitz and Subbarao [4], each of these identities can be proved in an elementary manner using the triple product identity. A comprehensive survey of the quintuple product identity can be found in [7].

Let us now define V_j as

$$V_j := \frac{\theta_1(z)\theta_2(z)\theta_3(z)\theta_4(z)}{\theta_j(z)K(q)}. \tag{2.12}$$

By expanding each V_j in z and switching the order of summation, we can obtain the following expression in terms of $T_{j,k}$.

$$V_1 = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} T_{1,2k}\,, \tag{2.13a}$$

$$V_j = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(2k-1)!} T_{j,2k-1}\,, \qquad j = 2,3,4. \tag{2.13b}$$

We also need the next lemma relating our four families of Eisenstein series.

Lemma 2.2. *For all positive integers $k > 0$,*

$$E_{1,2k} = E_{2,2k} + E_{3,2k} + E_{4,2k}.$$

Proof. It suffices to show that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^n n^{2k-1} q^{2n}}{1-q^{2n}} + \sum_{n=1}^{\infty} \frac{(-1)^n n^{2k-1} q^{n}}{1-q^{2n}} + \sum_{n=1}^{\infty} \frac{n^{2k-1} q^{n}}{1-q^{2n}} \\
&= \sum_{n=1}^{\infty} \frac{(2n)^{2k-1} q^{4n}}{1-q^{4n}} - \sum_{n=1}^{\infty} \frac{(2n-1)^{2k-1} q^{4n-2}}{1-q^{4n-2}} + \sum_{n=1}^{\infty} \frac{2(2n)^{2k-1} q^{2n}}{1-q^{4n}} \\
&= \sum_{n=1}^{\infty} \frac{2(2n)^{2k-1} (q^{2n}+q^{4n})}{1-q^{4n}} - \sum_{n=1}^{\infty} \frac{(2n-1)^{2k-1} q^{4n-2}}{1-q^{4n-2}} - \sum_{n=1}^{\infty} \frac{(2n)^{2k-1} q^{4n}}{1-q^{4n}} \\
&= \sum_{n=1}^{\infty} \frac{2^{2k} n^{2k-1} q^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty} \frac{n^{2k-1} q^{2n}}{1-q^{2n}} \\
&= (2^{2k}-1) \sum_{n=1}^{\infty} \frac{n^{2k-1} q^{2n}}{1-q^{2n}}.
\end{aligned}$$

□

We record the following corollary although we do not need it for our purposes. We refer the reader to [14, p. 330] or [15, Cor. 4.1.1] for the connection between the functions f_j and the invariants of the Weierstrass $\wp$-function.

Corollary 2.3.

$$f_2 + f_3 + f_4 = 0. \tag{2.14}$$

The next lemma is a reformulation of Ramanujan's differential equations (1.4) in terms of the variable q.

Lemma 2.4.

$$q\frac{\mathrm{d}E_{1,2}}{\mathrm{d}q} = \frac{1}{6}\left(E_{1,2}^2 - E_{1,4}\right), \tag{2.15a}$$

$$q\frac{\mathrm{d}E_{1,4}}{\mathrm{d}q} = \frac{2}{3}\left(E_{1,2}E_{1,4} - E_{1,6}\right), \tag{2.15b}$$

$$q\frac{\mathrm{d}E_{1,6}}{\mathrm{d}q} = E_{1,2}E_{1,6} - E_{1,4}^2. \tag{2.15c}$$

Chan [5] derived the above result from equations (2.3a), (2.7a) and (2.13a). We shall now show that the counterparts of these three equations, namely (2.3b), (2.7b) and (2.13b) are sufficient to prove our main results.

Proof of Theorem 1.1. Substituting (2.3b) into (2.7b), we obtain

$$\sum_{k=1}^{\infty} \frac{(-1)^k z^{2k-1}}{(2k-1)!} S_{j,2k} = \left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} S_{j,2k} \right) \left(\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} E_{j,2k} z^{2k-1} \right).$$

Equating coefficients of z yields the following relation

$$S_{j,2k} = \sum_{m=1}^{k} \frac{(2k-1)! 2^{2m}(2^{2m}-1)B_{2m}}{(2k-2m)!(2m)!} S_{j,2k-2m} E_{j,2m}. \tag{2.16}$$

For $k = 1$, 2 and 3, we obtain

$$S_{j,2} = S_{j,0} E_{j,2}, \tag{2.17a}$$

$$\begin{aligned} S_{j,4} &= 3S_{j,2}E_{j,2} - 2S_{j,0}E_{j,4} \\ &= S_{j,0}\left(3E_{j,2}^2 - 2E_{j,4}\right), \end{aligned} \tag{2.17b}$$

$$\begin{aligned} S_{j,6} &= 5S_{j,4}E_{2,2} - 20S_{j,2}E_{j,4} + 16S_{j,0}E_{j,6} \\ &= S_{j,0}\left(15E_{j,2}^3 - 30E_{j,2}E_{j,4} + 16E_{j,6}\right). \end{aligned} \tag{2.17c}$$

Applying $4q\frac{\mathrm{d}}{\mathrm{d}q}$ to equation (2.17a) gives us

$$S_{j,4} = S_{j,2}E_{j,2} + 4S_{j,0}\left(q\frac{\mathrm{d}E_{j,2}}{\mathrm{d}q}\right). \tag{2.18}$$

Comparing with equation (2.17b) proves equation (1.13b),

$$q\frac{\mathrm{d}E_{j,2}}{\mathrm{d}q} = \frac{E_{j,2}^2 - E_{j,4}}{2}.$$

Now we apply $4q\frac{\mathrm{d}}{\mathrm{d}q}$ to equation (2.17b) and together with equation (2.17c) deduce equation (1.13c),

$$q\frac{\mathrm{d}E_{j,4}}{\mathrm{d}q} = 2\left(E_{j,2}E_{j,4} - E_{j,6}\right).$$

Next we let $\{j, r, s\}$ denote some permutation of $\{2, 3, 4\}$. When we differentiate V_j, we obtain

$$V_j' = V_j\left(\frac{\theta_1'}{\theta_1}(z) + \frac{\theta_r'}{\theta_r}(z) + \frac{\theta_s'}{\theta_s}(z)\right).$$

Using equation (2.13b) we get

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-2}}{(2k-2)!} T_{j,2k-1} = \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1} z^{2k-1}}{(2k-1)!} T_{j,2k-1} \right) \left(\frac{\theta_1'}{\theta_1}(z) + \frac{\theta_r'}{\theta_r}(z) + \frac{\theta_s'}{\theta_s}(z) \right).$$

Making use of identities (2.7a) and (2.7b), and equating coefficients of z yields the following relation

$$
\begin{aligned}
&T_{j,2k+1}\\
&= \frac{1}{2k}\sum_{m=1}^{k}\frac{(2k+1)!2^{2m}B_{2m}}{(2k-2m+1)!(2m)!}T_{j,2k-2m+1}\left(E_{1,2m}+(2^{2m}-1)(E_{r,2m}+E_{s,2m})\right)\\
&= \frac{1}{2k}\sum_{m=1}^{k}\frac{(2k+1)!2^{2m}B_{2m}}{(2k-2m+1)!(2m)!}T_{j,2k-2m+1}\left(2^{2m}E_{1,2m}-(2^{2m}-1)E_{j,2m}\right),
\end{aligned}
$$

where Lemma 2.2 was used in the last step. We write out explicitly the identities for k from 1 to 3 and simplify where necessary with equation (1.12) to get

$$T_{j,3} = T_{j,1}(9E_{j,2}-8f_j), \tag{2.19a}$$

$$
\begin{aligned}
T_{j,5} &= \frac{5}{3}T_{j,3}(9E_{j,2}-8f_j) - \frac{2}{3}T_{j,1}(16E_{1,4}-15E_{j,4})\\
&= \frac{1}{3}T_{j,1}\left(5(9E_{j,2}-8f_j)^2 - 2(16E_{1,4}-15E_{j,4})\right),
\end{aligned} \tag{2.19b}
$$

$$
\begin{aligned}
T_{j,7} &= \frac{7}{3}T_{j,5}(9E_{j,2}-8f_j) - \frac{28}{9}T_{j,3}(16E_{1,4}-15E_{j,4}) + \frac{16}{9}T_{j,1}(64E_{1,6}-63E_{j,6})\\
&= \frac{1}{9}T_{j,1}\Bigg(35(9E_{j,2}-8f_j)^3 - 42(9E_{j,2}-8f_j)(16E_{1,4}-15E_{j,4})\\
&\qquad + 16(64E_{1,6}-63E_{j,6})\Bigg).
\end{aligned} \tag{2.19c}
$$

If we apply $12q\frac{\mathrm{d}}{\mathrm{d}q}$ to equation (2.19a) and compare with equation (2.19b), we have the following identity

$$18\left(9q\frac{\mathrm{d}E_{j,2}}{\mathrm{d}q} - 8q\frac{\mathrm{d}f_j}{\mathrm{d}q}\right) = (9E_{j,2}-8f_j)^2 - 16E_{1,4} + 15E_{j,4}. \tag{2.20}$$

Since we already have an expression for $q\frac{\mathrm{d}E_{j,2}}{\mathrm{d}q}$, we can simplify the above equation if we are able to eliminate the $E_{1,4}$ term. To do this, we apply $12q\frac{\mathrm{d}}{\mathrm{d}q}$ to equation (1.12) and use Lemma 2.4 to obtain

$$
\begin{aligned}
12q\frac{\mathrm{d}f_j}{\mathrm{d}q} &= 9E_{j,2}^2 - 9E_{j,4} - E_{1,2}^2 + E_{1,4}\\
&= 4f_j(f_j+E_{1,2}) - 9E_{j,4} + E_{1,4}\\
&= 12f_jE_{j,2} - 4f_j^2 - 9E_{j,4} + E_{1,4}.
\end{aligned} \tag{2.21}
$$

(Equation (1.12) was used to simplify the last two steps.) Now eliminating $E_{1,4}$ in equations (2.20) and (2.21), we obtain our goal

$$q\frac{\mathrm{d}f_j}{\mathrm{d}q} = f_jE_{j,2} - E_{j,4}. \qquad \square$$

Remark. Theorem 1.1 could also be proved by parameterizing each Eisenstein series in terms of

$$Z = \theta_3^2(0) \qquad \text{and} \qquad X = \frac{\theta_2^4(0)}{\theta_3^4(0)}. \tag{2.22}$$

Parameterizations for our four families and an additional twelve families of Eisenstein series can be found in [9].

Proof of Theorem 1.2. If we substitute identity (1.13a) into equation (2.21), we obtain the interesting identity

$$4f_j^2 = E_{1,4} + 3E_{j,4}. \tag{2.23}$$

We now apply $q\frac{\mathrm{d}}{\mathrm{d}q}$ to this identity and simplify with differential equations (1.13a), (1.13c) and (2.15b). Then

$$8f_j^2E_{j,2} - 8f_jE_{j,4} = \frac{2}{3}(E_{1,2}E_{1,4} - E_{1,6}) + 6(E_{j,2}E_{j,4} - E_{j,6}). \tag{2.24}$$

We simplify the above with identity (2.23) to get

$$\begin{aligned} \frac{2}{3}E_{1,6} + 6E_{j,6} &= 8f_jE_{j,4} - \frac{2}{3}E_{1,4}(3E_{j,2} - E_{1,2}) \\ &= 8f_jE_{j,4} - \frac{4}{3}f_jE_{1,4}. \end{aligned} \tag{2.25}$$

We now have a relation between $E_{j,6}$ and $f_jE_{j,4}$. To obtain another relation, we apply $12q\frac{\mathrm{d}}{\mathrm{d}q}$ to equation (2.19b) and compare it with equation (2.19c). After some simplification, we arrive at the following intermediate step.

$$\begin{aligned} &18q\frac{\mathrm{d}}{\mathrm{d}q}(16E_{1,4} - 15E_{j,4}) + 4(64E_{1,6} - 63E_{j,6}) \\ &= (9E_{j,2} - 8f_j)\left(9(16E_{1,4} - 15E_{j,4}) - 5(9E_{j,2} - 8f_j)^2 + 90\left(9q\frac{\mathrm{d}E_{j,2}}{\mathrm{d}q} - 8q\frac{\mathrm{d}f_j}{\mathrm{d}q}\right)\right). \end{aligned}$$

Next we use identity (2.20) to simplify the right hand side of the above to get

$$\begin{aligned} &9q\frac{\mathrm{d}}{\mathrm{d}q}(16E_{1,4} - 15E_{j,4}) \\ &= 2(9E_{j,2} - 8f_j)(16E_{1,4} - 15E_{j,4}) - 2(64E_{1,6} - 63E_{j,6}). \end{aligned} \tag{2.26}$$

Finally we use the differential equations (1.13c), (2.15b) and identity (1.12) to arrive at

$$2E_{1,6} + 9E_{j,6} = 15f_jE_{j,4} - 4f_jE_{1,4}. \tag{2.27}$$

Putting equations (2.25) and (2.27) together proves Theorem 1.2 □

3. Applications

We conclude this article with several applications. In [24, p. 369] Ramanujan used his differential equations to prove two family of identities equivalent to

$$S_{1,2k+1} = q^{1/4}\prod_{r=1}^{\infty}(1-q^{2r})^3 \sum_{2\ell+4m+6n=2k} \alpha_{\ell,m,n}E_{1,2}^{\ell}E_{1,4}^{m}E_{1,6}^{n}, \tag{3.1a}$$

$$T_{1,2k} = q^{1/12}\prod_{r=1}^{\infty}(1-q^{2r}) \sum_{2\ell+4m+6n=2k} \beta_{\ell,m,n}E_{1,2}^{\ell}E_{1,4}^{m}E_{1,6}^{n}. \tag{3.1b}$$

These follow from a straight forward inductive argument using (2.5) and (2.9), as well as the infinite product representations

$$S_{1,1} = q^{1/4}\prod_{r=1}^{\infty}(1-q^{2r})^3 \quad \text{and} \quad T_{1,0} = q^{1/12}\prod_{r=1}^{\infty}(1-q^{2r}). \tag{3.2}$$

Now the quintuple product identity gives us the following infinite product representations.

$$T_{2,1} = q^{1/12} \prod_{r=1}^{\infty} \frac{(1-q^{2r})^5}{(1-q^{4r})^2}, \tag{3.3a}$$

$$T_{3,1} = 2q^{1/3} \prod_{r=1}^{\infty} \frac{(1-q^{4r})^2(1-q^r)^2}{(1-q^{2r})}, \tag{3.3b}$$

$$T_{4,1} = 2q^{1/3} \prod_{r=1}^{\infty} \frac{(1-q^{2r})^5}{(1-q^r)^2}. \tag{3.3c}$$

Identities (3.3a) and (3.3b) can be found in Ramanujan's notebooks [1, p. 114]. Together with Theorem 1.1 and (2.9) we can obtain the following analogues of (3.1), which appear to be new.

Corollary 3.1.

$$T_{2,2k+1} = \left(q^{1/12} \prod_{r=1}^{\infty} \frac{(1-q^{2r})^5}{(1-q^{4r})^2}\right) \sum_{2\ell+2m+4n=2k} \delta_{2,\ell,m,n} f_2^\ell E_{2,2}^m E_{2,4}^n, \tag{3.4a}$$

$$T_{3,2k+1} = \left(q^{1/3} \prod_{r=1}^{\infty} \frac{(1-q^{4r})^2(1-q^r)^2}{(1-q^{2r})}\right) \sum_{2\ell+2m+4n=2k} \delta_{3,\ell,m,n} f_3^\ell E_{3,2}^m E_{3,4}^n, \tag{3.4b}$$

$$T_{4,2k+1} = \left(q^{1/3} \prod_{r=1}^{\infty} \frac{(1-q^{2r})^5}{(1-q^r)^2}\right) \sum_{2\ell+2m+4n=2k} \delta_{4,\ell,m,n} f_4^\ell E_{4,2}^m E_{4,4}^n. \tag{3.4c}$$

We can also obtain the companion identities involving $S_{j,k}$.

Corollary 3.2.

$$S_{2,2k} = \left(q^{1/4} \prod_{r=1}^{\infty} \frac{(1-q^{4r})^2}{(1-q^{2r})}\right) \sum_{2\ell+2m+4n=2k} \gamma_{2,\ell,m,n} f_2^\ell E_{2,2}^m E_{2,4}^n, \tag{3.5a}$$

$$S_{3,2k} = \left(\prod_{r=1}^{\infty} \frac{(1-q^{2r})^5}{(1-q^{4r})^2(1-q^r)^2}\right) \sum_{2\ell+2m+4n=2k} \gamma_{3,\ell,m,n} f_3^\ell E_{3,2}^m E_{3,4}^n, \tag{3.5b}$$

$$S_{4,2k} = \left(\prod_{r=1}^{\infty} \frac{(1-q^r)^2}{(1-q^{2r})}\right) \sum_{2\ell+2m+4n=2k} \gamma_{4,\ell,m,n} f_4^\ell E_{4,2}^m E_{4,4}^n. \tag{3.5c}$$

Identity (3.5a) was observed by Hahn [13] while (3.5b) and (3.5c) were recorded by Huber [16].

Remark. Other analogues of identities (3.1) can be found in [6] where the infinite products are replaced by certain lacunary powers of Dedekind's eta function.

Our next application is to the Halphen system of differential equations

$$q\frac{\mathrm{d}}{\mathrm{d}q}(u_1+u_2) = u_1u_2, \tag{3.6a}$$

$$q\frac{\mathrm{d}}{\mathrm{d}q}(u_2+u_3) = u_2u_3, \tag{3.6b}$$

$$q\frac{\mathrm{d}}{\mathrm{d}q}(u_3+u_1) = u_3u_1. \tag{3.6c}$$

G.H. Halphen [14, p. 330] demonstrated that a system of solutions is given by $E_{2,2}, E_{3,2}$ and $E_{4,2}$. It is easy to check that

$$E_{2,2}(q) + E_{3,2}(q) = E_{2,2}(q^{\frac{1}{2}}) = \mathcal{P}(q), \tag{3.7a}$$

$$E_{2,4}(q) + E_{3,4}(q) = E_{2,4}(q^{\frac{1}{2}}) = \mathcal{Q}(q). \tag{3.7b}$$

Differentiating equation (3.7a) with respect to q, we observe from identities (1.8b) and (1.13b) that

$$\begin{aligned} q\frac{\mathrm{d}E_{2,2}(q)}{\mathrm{d}q} + q\frac{\mathrm{d}E_{3,2}(q)}{\mathrm{d}q} &= \frac{\mathcal{P}^2(q) - \mathcal{Q}(q)}{4} \\ &= \frac{1}{2}\left(\frac{E_{2,2}^2(q) - E_{2,4}(q)}{2} + \frac{E_{3,2}^2(q) - E_{3,4}(q)}{2}\right) + \frac{1}{2}E_{2,2}(q)E_{3,2}(q) \\ &= \frac{1}{2}\left(q\frac{\mathrm{d}E_{2,2}(q)}{\mathrm{d}q} + q\frac{\mathrm{d}E_{3,2}(q)}{\mathrm{d}q}\right) + \frac{1}{2}E_{2,2}(q)E_{3,2}(q). \end{aligned} \tag{3.8}$$

(3.6b) and (3.6c) can be done in a similar manner. For more information on equations (3.6), see [27, p. 57-76], [29], [11], [16] and [8].

Finally, J.W.L. Glaisher [10] defined seven divisor sums and listed various relations among them. These divisor sums can be written as the coefficients of Eisenstein series. For example, if we define

$$\sigma_s(n) = \sum_{d|n} d^s, \tag{3.9a}$$

$$\widetilde{\sigma}_s(n) = \sum_{d|n} (-1)^{d-1} d^s, \tag{3.9b}$$

$$\widehat{\sigma}_s(n) = \sum_{d|n} (-1)^{\frac{n}{d}-1} d^s, \tag{3.9c}$$

for positive integers n and s, we can rewrite some of the previously defined Eisenstein series as

$$Q(x) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) x^n, \tag{3.10a}$$

$$\mathcal{Q}(x) = 1 - 16 \sum_{n=1}^{\infty} \widetilde{\sigma}_3(n) x^n, \tag{3.10b}$$

$$\mathcal{E}(x) = 1 + 24 \sum_{n=1}^{\infty} \widehat{\sigma}_1(n) x^n. \tag{3.10c}$$

Thus whenever we have an identity or a differential equation involving Eisenstein series, a corresponding identity involving these divisor sums can be written down. Hahn [12] recently established several such convolution sum identities, one of which is the following.

$$36 \sum_{m<n} \widehat{\sigma}_1(n)\widehat{\sigma}_1(n-m) = \begin{cases} -3\widehat{\sigma}_1(n) + 3\widetilde{\sigma}_3(n), & \text{if } n \text{ is odd,} \\ -3\widehat{\sigma}_1(n) - 5\widetilde{\sigma}_3(n) + 4\widetilde{\sigma}_3(n/2), & \text{if } n \text{ is even.} \end{cases} \tag{3.11}$$

Now as a corollary of identity (2.23) for $j = 2$, we have the following result.

Corollary 3.3.

$$48 \sum_{m<n} \widehat{\sigma}_1(n)\widehat{\sigma}_1(n-m) = -4\widehat{\sigma}_1(n) + 5\sigma_3(n) - \widetilde{\sigma}_3(n).$$

Many analogous results, including those corresponding to Eisenstein series $E_{3,2k}$ and $E_{4,2k}$ can be derived from the identities proved in this article.

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References

[1] Bruce C. Berndt, *Ramanujan's notebooks. Part III,* Springer-Verlag, New York, 1991.

[2] B.C. Berndt, *Number theory in the spirit of Ramanujan*, Amer. Math. Soc., Providence, Rhode Island, 2006.

[3] B.C. Berndt, P. Bialek, A.J. Yee, *Formulas of Ramanujan for the power series coefficients of certain quotients of Eisenstein series*, Int. Math. Res. Notices, **21** (2002), 1077–1109.

[4] L. Carlitz, M. V. Subbarao, *A simple proof of the quintuple product identity,* Proc. Amer. Math. Soc. **32** (1972) 42-44.

[5] H.H. Chan, *Triple product identity, quintuple product identity and Ramanujan's differential equations for the classical Eisenstein series,* Proc. Amer. Math. Soc. **135** (2007) 1987–1992.

[6] H.H. Chan, S. Cooper, P.C. Toh *Ramanujan's Eisenstein series and powers of Dedekind's eta-function,* J. London. Math. Soc. **75** (2007) 225–242.

[7] S. Cooper, *The quintuple product identity*, Int. J. Number Theory, **2** (2006), 115–161.

[8] S. Cooper, *A review of Venkatachaliengar's book on elliptic functions,* preprint.

[9] S. Cooper, H. Y. Lam, *Sixteen Eisenstein series,* Ramanujan J., **18** (2009), 33–59.

[10] J.W.L. Glaisher, *On certain sums of products quantities depending upon the divisors of a number,* Mess. Math., **15** (1885), 1–20.

[11] H. Hahn, *Eisenstein series, analogues of the Rogers-Ramanujan functions, and partition identities,* Ph.D. Thesis, University of Illinois at Urbana-Champaign, 2004.

[12] H. Hahn, *Convolution sums of some functions on divisors,* Rocky Mountain J. Math., **37** (2007), 1593–1622.

[13] H. Hahn, *Eisenstein series associated with* $\Gamma_0(2)$*,* Ramanujan J., **15** (2008), 235–257.

[14] G.H. Halphen, *Traité des fonctions elliptiques et de leurs applications,* Part 1, Gauthier-Villars, Paris, 1886.

[15] T. Huber, *Zeros of generalized Rogers-Ramanujan series and topics from Ramanujan's theory of elliptic functions*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 2007.

[16] T. Huber, *Coupled systems of differential equations for modular forms of level* n, preprint.

[17] N. Koblitz, *Introduction to elliptic curves and modular forms*, 2nd ed., Springer, New York 1993.

[18] Z.-G. Liu, *A three-term theta function identity and its applications*, Adv. Math., **195** (2005), 1-23.

[19] I.G. Macdonald, *Affine root systems and Dedekind's* η*-function*, Invent. Math., **15** (1972), 91-143.

[20] R.S. Maier, *Nonlinear differential equations satisfied by certain classical modular forms*, arXiv:0807.1081v2, 8 Jul. 2008.

[21] V. Ramamani, *Some identities conjectured by Srinivasa Ramanujan in his lithographed notes connected with partition theory and elliptic modular functions - their proofs - interconnections with various other topics in the theory of numbers and some generalizations thereon,* Ph.D. Thesis, University of Mysore, 1970.

[22] V. Ramamani, *On some algebraic identities connected with Ramanujan's work,* in: Ramanujan International Symposium on Analysis (Pune, 1987), Macmillan of India, New Delhi, (1989), 277-291.

[23] S. Ramanujan, *On certain arithmetical functions,* Trans. Cambridge Philos. Soc., **22** (1916), 159–184.

[24] S. Ramanujan, *The lost notebook and other unpublished papers*, Narosa, New Delhi, 1988.

[25] L.-C. Shen, *On the products of three theta functions,* Ramanujan J., **3** (1999), 343–357.

[26] P. C. Toh, *Generalized mth order Jacobi theta functions and the Macdonald identities,* Int. J. Number Theory, **4** (2008), 461–474.

[27] K. Venkatachaliengar, *Development of elliptic functions according to Ramanujan,* Technical Report 2, Madurai Kamaraj University, Department of Mathematics, Madurai, 1988.

[28] E.T. Whittaker, G.N. Watson, *A course of modern analysis,* 4th ed., Cambridge University Press, Cambridge, 1927.

[29] W. Zudilin, *The hypergeometric equation and Ramanujan functions,* Ramanujan J., **7** (2003), 435–447.

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