Generalized m-th Order Jacobi Theta Functions And The Macdonald Identities

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Abstract

We describe a m-th order generalization of Jacobi's theta functions and use these functions to construct classes of theta function identities in multiple variables. These identities are equivalent to the Macdonald identities for the seven infinite families of irreducible affine root systems. They are also equivalent to some elliptic determinant evaluations proven recently by H. Rosengren and M. Schlosser.

Keywords: determinants, theta functions, elliptic functions, Macdonald identities
2000 Mathematics Subject Classification: 11F20, 14K25, 15A15, 33D52, 33E05

1 Introduction

The quintuple product identity

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}(a^{3n-1}+a^{-3n}) = \prod_{n=1}^{\infty}(1-q^n)(1+aq^n)(1+a^{-1}q^{n-1})(1-a^2q^{2n-1})(1-a^{-2}q^{2n-1}), \tag{1.1}$$

is an important identity with a very rich history. Beginning from G.N. Watson in 1929 [22] till the present, many different proofs of this identity have been published. Surveys of the various proofs of the quintuple product identity can be found in [1, p.83] and [6].

The Hirschhorn-Farkas-Kra septagonal numbers identity,

$$\begin{aligned}\sum_{m=-\infty}^{\infty}(-1)^m q^{(5m^2-m)/2}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(5n^2-3n)/2}(a^{5n-3}+a^{-5n})\right)& \quad (1.2)\\ -\sum_{m=-\infty}^{\infty}(-1)^m q^{(5m^2-3m)/2}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(5n^2-n)/2}(a^{5n-2}+a^{-5n-1})\right)&\\ =\prod_{n=1}^{\infty}(1-q^n)^2(1-aq^n)(1-a^{-1}q^{n-1})(1-a^2q^n)(1-a^{-2}q^{n-1}).&\end{aligned}$$

is an analogue of the quintuple product. It was first discovered by M.D. Hirschhorn [11] and later rediscovered independently by H.M. Farkas and I. Kra [8]. In their important work, Farkas and Kra approached the problem from a function theoretic perspective and developed an entire theory of function spaces of N-th order θ-function (with rational characteristics) [8], [9, p.129-145, p.268-273].

More recently, H.H. Chan, Z.-G. Liu and S.T. Ng [3] used the theory of elliptic functions to give elegant proofs of (1.1) and (1.2). In this present article, we generalize the idea used in [3] to derive classes of identities involving a m-th order generalization of the classical Jacobi theta functions. These are essentially identical to the functions used by Farkas and Kra, but our approach has a different flavour. Moreover, these identities involve multiple variables and are stronger than those derived by Farkas and Kra. As an illustration, (1.1) and (1.2) are special cases of the following,

$$\det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^\ell q^{(2n+1)\ell^2+(2j-1)\ell}\left(e^{(4n\ell+2\ell+2j-1)iz_k}+e^{-(4n\ell+2\ell+2j-1)iz_k}\right)\right)$$
$$=\frac{q^{-\frac{n^2}{4}}}{(q^2)_\infty^{n^2+n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)\prod_{\ell=1}^{n}\theta_2(z_\ell|\tau)\theta_3(z_\ell|\tau)\theta_4(z_\ell|\tau),$$

with $n=1$ and 2 respectively.

The infinite product on the right hand side of the above identity turns out to be equivalent to the infinite product that appears in the Macdonald identity for the affine root system BC_n [17].

This development lead to a literature search which revealed that our identities have been anticipated by H. Rosengren and M. Schlosser [19]. Although the results and proofs are very similar, the approaches to constructing these identities are different. Rosengren and Schlosser approached the problem from a determinant evaluation point of view. Their aim was to study the elliptic analogues of the Weyl denominator formulas and they used the affine root systems as their starting point, defining a special theta function for each affine root system.

Our approach, on the other hand, was to study identities constructed from m-th order theta functions. This construction does not make use of the affine root systems in any way. For each m, there are essentially four different theta functions, each satisfying a different transformation formula. (See (2.1) for the four transformations satisfied by the classical Jacobi theta functions. This is the case where $m = 1$.) By grouping the m-th order theta functions into odd and even functions, and considering the parity of m, we end up with sixteen classes of identities. Each of these identities has an infinite product corresponding to one of the seven infinite families of irreducible affine root systems described by Macdonald [17]. In a sense, our work can be viewed as an elementary approach to the Macdonald identities that is independent of root systems. For another elementary approach to the Macdonald identities for the infinite families, see [21].

In Section 2, we will define the classical Jacobi theta functions and list some of their properties which are crucial in this work. Then we will describe a m-th order generalization of the theta functions. In Section 3, we will use this m-th order theta function to construct classes of identities equivalent to the Macdonald identities. Examples of well known identities which occur as special cases of the main theorems, as well as some new formulas for powers of $(q)_\infty$ will be given in Section 4.

2 Jacobi Theta Functions

Let $q = e^{\pi i \tau}$ where $\mathrm{Im}(\tau) > 0$. The classical Jacobi theta functions are defined by

$$\begin{aligned} \theta_1(z|\tau) &= -iq^{\frac{1}{4}} \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2+k} e^{(2k+1)iz}, \\ \theta_2(z|\tau) &= q^{\frac{1}{4}} \sum_{k=-\infty}^{\infty} q^{k^2+k} e^{(2k+1)iz}, \\ \theta_3(z|\tau) &= \sum_{k=-\infty}^{\infty} q^{k^2} e^{2kiz} \\ \text{and} \qquad \theta_4(z|\tau) &= \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2kiz}. \end{aligned}$$

Jacobi's theta functions are functions of one complex variable z and a parameter τ. Throughout this article, we do not explore the "modular" properties of the theta functions and all identities can be considered as formal q-series identities. We list some important properties, the proofs of which can all be found in [23, Ch. 21].

The Jacobi theta functions satisfies the following transformation:

$$
\begin{aligned}
\theta_1(z+\pi|\tau) &= -\theta_1(z|\tau), & \theta_1(z+\pi\tau|\tau) &= -q^{-1}e^{-2iz}\theta_1(z|\tau), \\
\theta_2(z+\pi|\tau) &= -\theta_2(z|\tau), & \theta_2(z+\pi\tau|\tau) &= \ \ q^{-1}e^{-2iz}\theta_2(z|\tau), \\
\theta_3(z+\pi|\tau) &= \ \ \theta_3(z|\tau), & \theta_3(z+\pi\tau|\tau) &= \ \ q^{-1}e^{-2iz}\theta_3(z|\tau), \\
\theta_4(z+\pi|\tau) &= \ \ \theta_4(z|\tau) \quad \text{and} & \theta_4(z+\pi\tau|\tau) &= -q^{-1}e^{-2iz}\theta_4(z|\tau).
\end{aligned} \tag{2.1}
$$

Hence they are quasi-elliptic with (quasi) periods π and $\pi\tau$ and it suffices to study their values in the fundamental parallelogram,

$$\Pi = \{a\pi + b\pi\tau \mid 0 \le a < 1,\ 0 \le b < 1\}.$$

Each θ_i vanishes at exactly one point in Π and we have [23, p.465]

$$\theta_1(0|\tau) = \theta_2\Big(\frac{\pi}{2}\Big|\tau\Big) = \theta_3\Big(\frac{\pi+\pi\tau}{2}\Big|\tau\Big) = \theta_4\Big(\frac{\pi\tau}{2}\Big|\tau\Big) = 0.$$

Furthermore, each θ_i can be expressed as infinite products. We adopt the following notation:

$$(a_1,\dots,a_k;q)_\infty = \prod_{n=1}^{\infty}(1-a_1q^{n-1})\dots(1-a_kq^{n-1}) \qquad \text{and} \qquad (q)_\infty = \prod_{n=1}^{\infty}(1-q^n),$$

and note that [23, p.469]

$$
\begin{aligned}
\theta_1(z|\tau) &= 2q^{\frac14}\sin z(q^2;q^2)_\infty(q^2e^{2iz};q^2)_\infty(q^2e^{-2iz};q^2)_\infty, \\
\theta_2(z|\tau) &= 2q^{\frac14}\cos z(q^2;q^2)_\infty(-q^2e^{2iz};q^2)_\infty(-q^2e^{-2iz};q^2)_\infty, \\
\theta_3(z|\tau) &= (q^2;q^2)_\infty(-qe^{2iz};q^2)_\infty(-qe^{-2iz};q^2)_\infty, \\
\theta_4(z|\tau) &= (q^2;q^2)_\infty(qe^{2iz};q^2)_\infty(qe^{-2iz};q^2)_\infty.
\end{aligned} \tag{2.2}
$$

We now construct a m-th order generalization of Jacobi's theta functions.

Definition 2.1 *Let $m, j \in \mathbb{Z}$ and $l = 0$ or 1, define*

$$
\begin{aligned}
&T^0_{m,j}(z) = \sum_{k=-\infty}^{\infty} q^{mk^2+jk}e^{(2mk+j)iz}, \quad T^1_{m,j}(z) = \sum_{k=-\infty}^{\infty}(-1)^k q^{mk^2+jk}e^{(2mk+j)iz}, \\
&E^l_{m,j}(z) = T^l_{m,j}(z) + T^l_{m,j}(-z) \qquad \textit{and} \qquad O^l_{m,j}(z) = T^l_{m,j}(z) - T^l_{m,j}(-z).
\end{aligned}
$$

It is easy to check that the following identities hold,

$$T^l_{m,j}(z) = T^l_{m,-j}(-z) \qquad \text{and} \qquad T^l_{m,j}(z) = (-1)^l q^{m+j}T^l_{m,2m+j}(z).$$

As an immediate consequence,

$$O^0_{m,0}(z) \equiv O^1_{m,0}(z) \equiv O^0_{m,m}(z) \equiv E^1_{m,m}(z) \equiv 0. \tag{2.3}$$

Definition 2.2 *Let m be fixed. Define $V^l_{m,j}$ to be the complex vector space consisting of entire functions, $F^l_{m,j}(z)$, satisfying the following transformation formula:*

$$F^l_{m,j}(z+\pi) = (-1)^j F^l_{m,j}(z) \quad \text{and} \quad F^l_{m,j}(z+\pi\tau) = (-1)^l q^{-m} e^{-2miz} F^l_{m,j}(z). \tag{2.4}$$

It is clear that for each m, there are only four distinct spaces, determined by the parity of l and j. We can check that the functions $T^l_{m,j}(z)$, $E^l_{m,j}(z)$ and $O^l_{m,j}(z)$ all belong to $V^l_{m,k}$ whenever $j \equiv k \pmod 2$.

Lemma 2.3 *$V^l_{m,k}$ has dimension m and the set of functions*

$$\left\{T^l_{m,j}(z) \mid j \equiv k \pmod 2, -m < j \leq m\right\}$$

forms a basis.

Proof. Follow the treatment in [9, p.129-135]. □

Lemma 2.4 *Each element of $V^l_{m,k}$ has exactly m zeroes in Π, the fundamental parallelogram.*

Proof. Follow the method in [23, p.465]. □

The following corollary of the transformation formula (2.4) is useful for locating zeroes of $F^l_{m,j}(z)$.

Corollary 2.5 *The functions $E^l_{m,j}(z)$ and $O^l_{m,j}(z)$ have the following special values:*

$$F^l_{m,j}\left(\frac{\pi+\pi\tau}{2}\right) = (-1)^{j+m+l} F^l_{m,j}\left(-\frac{\pi+\pi\tau}{2}\right),$$
$$F^l_{m,j}\left(\frac{\pi}{2}\right) = (-1)^j F^l_{m,j}\left(-\frac{\pi}{2}\right) \qquad \text{and} \qquad F^l_{m,j}\left(\frac{\pi\tau}{2}\right) = (-1)^l F^l_{m,j}\left(-\frac{\pi\tau}{2}\right).$$

3 Main Identities

We have seen that for each m, we have four vector spaces $V^l_{m,j}$. We can further decompose $V^l_{m,j}$ into subspaces of odd and even functions with $\{O^l_{m,j}(z)\}$ and $\{E^l_{m,j}(z)\}$ as the respective basis elements. The dimensions of the odd and even subspaces now also depend on the parity of m, and can be tabulated with the help of (2.3).

Basis	m odd		m even	
Elements	j odd	j even	j odd	j even
$O^0_{m,j}(z)$	$\frac{m-1}{2}$	$\frac{m-1}{2}$	$\frac{m}{2}$	$\frac{m}{2}-1$
$E^0_{m,j}(z)$	$\frac{m+1}{2}$	$\frac{m+1}{2}$	$\frac{m}{2}$	$\frac{m}{2}+1$
$O^1_{m,j}(z)$	$\frac{m+1}{2}$	$\frac{m-1}{2}$	$\frac{m}{2}$	$\frac{m}{2}$
$E^1_{m,j}(z)$	$\frac{m-1}{2}$	$\frac{m+1}{2}$	$\frac{m}{2}$	$\frac{m}{2}$

Dimensions of odd and even subspaces of $V^l_{m,j}$

For example, consider the space $V^0_{6,2}$. Since both m and j are even, this space can be decomposed into an odd subspace of dimension 2, with basis $\{O^0_{6,2}(z), O^0_{6,4}(z)\}$ and an even subspace of dimension 4, with basis $\{E^0_{6,0}(z)$, $E^0_{6,2}(z)$, $E^0_{6,4}(z)$, $E^0_{6,6}(z)\}$.

For each of these sixteen subspaces, we now construct a multi-variable theta function identity with the basis elements. The most natural way to combine these elements symmetrically into an equation is via the determinant function. Once a choice of determinant function is written down, the right hand side of the identity is determined by the location of the zeroes that appear in the fundamental parallelogram, Π. To simplify notation, we let n denote the number of basis elements.

Theorem 3.1 *Let z_j be arbitrary complex variables, then the following identities hold:*

$$\det_{1\le j,k\le n}\left(E^1_{2n-1,2j-2}(z_k)\right) = \frac{2q^{-\frac{n^2-n}{4}}}{(q^2)^{n^2-n}_\infty}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)\prod_{\ell=1}^{n}\theta_4(z_\ell|\tau). \tag{3.1}$$

$$\det_{1\le j,k\le n}\left(E^1_{2n+1,2j-1}(z_k)\right) = \frac{q^{-\frac{n^2}{4}}}{(q^2)^{n^2+n}_\infty}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau) \times\prod_{\ell=1}^{n}\theta_2(z_\ell|\tau)\theta_3(z_\ell|\tau)\theta_4(z_\ell|\tau). \tag{3.2}$$

$$\det_{1\le j,k\le n}\left(E^1_{2n,2j-2}(z_k)\right) = \frac{2q^{-\frac{n^2-n}{4}}(q^4)_\infty}{(q^2)^{n^2+1}_\infty}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau) \times\prod_{\ell=1}^{n}\theta_3(z_\ell|\tau)\theta_4(z_\ell|\tau). \tag{3.3}$$

$$\det_{1\le j,k\le n}\left(E^1_{2n,2j-1}(z_k)\right) = \frac{q^{-\frac{n^2}{4}}}{(q)_\infty(q^2)^{n^2-2}_\infty(q^4)_\infty}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau) \times\prod_{\ell=1}^{n}\theta_2(z_\ell|\tau)\theta_4(z_\ell|\tau). \tag{3.4}$$

$$\det_{1\le j,k\le n}\left(O^1_{2n-1,2j-1}(z_k)\right) = \frac{i^n 2q^{-\frac{n^2}{4}}}{(q^2)^{n^2-n}_\infty}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)\prod_{\ell=1}^{n}\theta_1(z_\ell|\tau). \tag{3.5}$$

$$\det_{1\le j,k\le n}\left(O^1_{2n+1,2j}(z_k)\right)=\frac{i^n q^{-\frac{n^2+n}{4}}}{(q^2)_\infty^{n^2+n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)$$
$$\times\prod_{\ell=1}^{n}\theta_1(z_\ell|\tau)\theta_2(z_\ell|\tau)\theta_3(z_\ell|\tau). \tag{3.6}$$

$$\det_{1\le j,k\le n}\left(O^1_{2n,2j}(z_k)\right)=\frac{i^n 2q^{-\frac{n^2+n}{4}}(q^4)_\infty}{(q^2)_\infty^{n^2+1}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)$$
$$\times\prod_{\ell=1}^{n}\theta_1(z_\ell|\tau)\theta_2(z_\ell|\tau). \tag{3.7}$$

$$\det_{1\le j,k\le n}\left(O^1_{2n,2j-1}(z_k)\right)=\frac{i^n q^{-\frac{n^2}{4}}}{(q)_\infty(q^2)_\infty^{n^2-2}(q^4)_\infty}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)$$
$$\times\prod_{\ell=1}^{n}\theta_1(z_\ell|\tau)\theta_3(z_\ell|\tau). \tag{3.8}$$

$$\det_{1\le j,k\le n}\left(E^0_{2n-1,2j-2}(z_k)\right)=\frac{2q^{-\frac{n^2-n}{4}}}{(q^2)_\infty^{n^2-n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)\prod_{\ell=1}^{n}\theta_3(z_\ell|\tau). \tag{3.9}$$

$$\det_{1\le j,k\le n}\left(E^0_{2n-1,2j-1}(z_k)\right)=\frac{2q^{-\frac{n^2}{4}}}{(q^2)_\infty^{n^2-n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)\prod_{\ell=1}^{n}\theta_2(z_\ell|\tau) \tag{3.10}$$

$$\det_{1\le j,k\le n}\left(E^0_{2n-2,2j-2}(z_k)\right)=\frac{4q^{-\frac{n^2-n}{4}}}{(q^2)_\infty^{n^2-2n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau). \tag{3.11}$$

$$\det_{1\le j,k\le n}\left(E^0_{2n,2j-1}(z_k)\right)=\frac{q^{-\frac{n^2}{4}}}{(-q;q^2)_\infty(q^2)_\infty^{n^2-1}(q^4)_\infty}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)$$
$$\times\prod_{\ell=1}^{n}\theta_2(z_\ell|\tau)\theta_3(z_\ell|\tau). \tag{3.12}$$

$$\det_{1\le j,k\le n}\left(O^0_{2n+1,2j-1}(z_k)\right)=\frac{i^n q^{-\frac{n^2}{4}}}{(q^2)_\infty^{n^2+n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)$$
$$\times\prod_{\ell=1}^{n}\theta_1(z_\ell|\tau)\theta_3(z_\ell|\tau)\theta_4(z_\ell|\tau). \tag{3.13}$$

$$\det_{1\le j,k\le n}\left(O^0_{2n+1,2j}(z_k)\right) = \frac{i^n q^{-\frac{n^2+n}{4}}}{(q^2)_\infty^{n^2+n}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|\tau)$$
$$\times \prod_{\ell=1}^{n} \theta_1(z_\ell|\tau)\theta_2(z_\ell|\tau)\theta_4(z_\ell|\tau). \tag{3.14}$$

$$\det_{1\le j,k\le n}\left(O^0_{2n+2,2j}(z_k)\right) = \frac{i^n q^{-\frac{n^2+n}{4}}}{(q^2)_\infty^{n^2+2n}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|\tau)$$
$$\times \prod_{\ell=1}^{n} \theta_1(z_\ell|\tau)\theta_2(z_\ell|\tau)\theta_3(z_\ell|\tau)\theta_4(z_\ell|\tau). \tag{3.15}$$

$$\det_{1\le j,k\le n}\left(O^0_{2n,2j-1}(z_k)\right) = \frac{i^n q^{-\frac{n^2}{4}}}{(-q;q^2)_\infty (q^2)_\infty^{n^2-1}(q^4)_\infty} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|\tau)$$
$$\times \prod_{\ell=1}^{n} \theta_1(z_\ell|\tau)\theta_4(z_\ell|\tau). \tag{3.16}$$

We provide a few remarks before giving the proof of one case in detail. The other cases will follow from similar arguments. First of all, since all the four $\theta_i(z|\tau)$ are equivalent up to a half period transform [23, p. 464], some of the above identities are equivalent. For example (3.5) can be obtained from (3.1) by replacing z_j with $z_j + \pi\tau/2$ for all j.

Secondly, the infinite product on the right hand side of each identity corresponds to an affine root system. In fact, these identities are equivalent to the Macdonald identities [17]. Specifically, we have:

Root System	Identity			
B_n	(3.5)	(3.1)	(3.9)	(3.10)
$B_n^\vee$	(3.7)	(3.3)		
BC_n	(3.13)	(3.2)	(3.6)	(3.14)
C_n	(3.15)			
$C_n^\vee$	(3.16)	(3.4)	(3.8)	(3.12)
D_n	(3.11)			

Identities appearing in the same row are equivalent and the identity in the first column is the one that is most easily recognized from the affine root system. The identities for A_n will be discussed in Theorem 3.2.

The third remark is that all of the above theorems have been independently discovered by Rosengren and Schlosser [19, Prop 3.4]. They have also shown in their paper, the equivalence of these determinant identities to the Macdonald identities.

Proof of Identity (3.15)**.**

Let $F(z_1, \ldots, z_n)$ denote the determinant expression in (3.15). We first assume that $\{z_j | j \neq 1\}$ are fixed, distinct complex numbers in the fundamental parallelogram Π, that are different from $0, \frac{\pi}{2}, \frac{\pi\tau}{2}$ and $\frac{\pi+\pi\tau}{2}$. Then, $F(z_1, \ldots, z_n)$ can be considered as a function of z_1, i.e $F(z_1, \ldots, z_n) = F(z_1)$. Since it is a linear combination of $O^0_{2n+2,2j}(z_1)$ for $j = 1$ to n, it satisfies the transformation formula (2.4).

Moreover as a function of z_1, $F(z_1)$ is odd. Corollary 2.5 allows us to conclude that $F(z_1) = 0$ at each of the four values $0, \frac{\pi}{2}, \frac{\pi\tau}{2}$ and $\frac{\pi+\pi\tau}{2}$. It is also evident that $F(\pm z_j) = 0$, $2 \leq j \leq n$. This accounts for all the $2n+2$ zeroes of $F(z_1)$ in Π. (The points $-z_j$ are not in Π but their equivalent points $-z_j+\pi+\pi\tau$ are.)

Now, let

$$P(z_1, \ldots, z_n) = \prod_{1 \leq j < k \leq n} \theta_1(z_j \pm z_k|\tau) \prod_{\ell=1}^{n} \theta_1(z_\ell|\tau)\theta_2(z_\ell|\tau)\theta_3(z_\ell|\tau)\theta_4(z_\ell|\tau).$$

From the formulas in (2.1), we can see that as a function of z_1, i.e. $P(z_1, \ldots, z_n) = P(z_1)$ satisfies the same transformation formula as $F(z_1)$, and has the same zeroes. Thus the quotient $F(z_1)/P(z_1)$ is elliptic and entire. Appealing to Liouville's theorem, $F(z_1)/P(z_1)$ is a "constant" expression $c(z_2, \ldots, z_n, \tau)$ that is independent of z_1.

We can repeat the same argument for each of the z_j and conclude that the quotient $F(z_1, \ldots, z_n)/P(z_1, \ldots, z_n)$ equals a constant $c(\tau)$ that is dependent only on τ. The principle of analytic continuation then allows us to conclude that the identity holds for all z_j.

We now calculate $c(\tau)$. Let $z_k = \pi k/(2n+2)$ and let w denote the primitive $(2n+2)$-th root of unity, i.e.

$$w^k = e^{2iz_k} = e^{\frac{2\pi i}{2n+2}k}.$$

Thus,

$$\begin{aligned}
&\det_{1 \leq j,k \leq n} \left(O^0_{2n+2,2j}(z_k)\right) \\
&= \det_{1 \leq j,k \leq n} \left(\sum_{\ell=-\infty}^{\infty} q^{(2n+2)\ell^2+2j\ell}(w^{kj} - w^{-kj}) \right) \\
&= \left(\prod_{j=1}^{n} (q^{4n+4}, -q^{2n+2+2j}, -q^{2n+2-2j}; q^{4n+4})_\infty \right) \det_{1 \leq j,k \leq n} \left(w^{kj} - w^{-kj}\right) \\
&= \left((q^{4n+4})^n_\infty \prod_{j=1}^{2n+2} (-q^{2j}; q^{4n+4})_\infty \right) (-q^{2n+2}; q^{2n+2})^{-1}_\infty \det_{1 \leq j,k \leq n} \left(w^{kj} - w^{-kj}\right) \\
&= (q^{4n+4})^{n-1}_\infty (q^{2n+2}; q^{2n+2})_\infty (-q^2; q^2)_\infty \det_{1 \leq j,k \leq n} \left(w^{kj} - w^{-kj}\right). \qquad (3.17)
\end{aligned}$$

where we have used the Jacobi triple product (2.2) to convert the series into infinite products.

To evaluate the determinant expression explicitly, we use [12, identity (2.3)] to obtain

$$\begin{aligned}
&\det_{1\le j,k\le n}\left(w^{kj}-w^{-kj}\right)\\
&= (w^1\dots w^n)^{-n}\prod_{1\le j<k\le n}(w^j-w^k)(1-w^jw^k)\prod_{k=1}^{n}(w^{2k}-1)\\
&= (w^1\dots w^n)^{-n+1}\prod_{1\le j<k\le n}(w^j-w^k)(1-w^jw^k)\prod_{k=1}^{n}(w^k-w^{-k})\\
&= \prod_{1\le j<k\le n}(w^{\frac{j-k}{2}}-w^{\frac{k-j}{2}})(w^{\frac{-j-k}{2}}-w^{\frac{j+k}{2}})\prod_{k=1}^{n}(w^k-w^{-k}). \qquad (3.18)
\end{aligned}$$

Next, we evaluate the right hand side of (3.15) for the same values of z_k. We use the following identity [23, p. 488],

$$\theta_1(z|\tau)\theta_2(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau)=q^{\frac{1}{4}}(q^2)_\infty^3\theta_1(2z|\tau),$$

and the infinite product formulas (2.2) to obtain

$$\begin{aligned}
&\prod_{\ell=1}^{n}\theta_1(z_\ell|\tau)\theta_2(z_\ell|\tau)\theta_3(z_\ell|\tau)\theta_4(z_\ell|\tau)\\
&= (-i)^nq^{\frac{n}{2}}(q^2)_\infty^{4n}\prod_{\ell=1}^{n}(w^\ell-w^{-\ell})(q^2w^{2\ell},q^2w^{-2\ell};q^2)_\infty\\
&= (-i)^nq^{\frac{n}{2}}(q^2)_\infty^{4n-2}(q^{2n+2};q^{2n+2})_\infty^2\prod_{\ell=1}^{n}(w^\ell-w^{-\ell}), \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|\tau)\\
&= (-i)^{n^2-n}q^{\frac{n^2-n}{4}}(q^2)_\infty^{n^2-n}\prod_{1\le j<k\le n}(w^{\frac{j-k}{2}}-w^{\frac{k-j}{2}})(w^{\frac{j+k}{2}}-w^{\frac{-j-k}{2}})\\
&\qquad\times\prod_{1\le j<k\le n}(q^2w^{j-k},q^2w^{k-j},q^2w^{j+k},q^2w^{-j-k};q^2)_\infty. \qquad (3.20)
\end{aligned}$$

We first observe that the products involving only powers of w is equal to the determinant evaluation in (3.18) up to a factor of $(-1)^{\frac{n^2-n}{2}}$. To simplify the four infinite products, we set k as j and j as k for the second product, k as $k+1$

for the third product, j as $n-j$ and k as $n-k+1$ for the last product to get

$$
\begin{aligned}
&\prod_{1\le j<k\le n} (q^2w^{j-k}, q^2w^{k-j}, q^2w^{j+k}, q^2w^{-j-k}; q^2)_\infty.\\
= \quad & \frac{\prod_{j=1}^n \prod_{k=1}^n (q^2w^{j-k}; q^2)_\infty}{\prod_{k=1}^n (q^2)_\infty} \times \prod_{j=1}^{n-1}\prod_{k=1}^{n-1} (q^2w^{j+k+1}; q^2)_\infty \prod_{k=1}^{n-1} (q^2w^{2k+1}; q^2)_\infty\\
= \quad & \frac{\prod_{j=1}^{n-1} \prod_{k=1}^{2n+2} (q^2w^{j+k}; q^2)_\infty}{(q^2)_\infty^{n-1} \prod_{k=1}^{n-1} (q^2w^{k+1}, q^2w^{n+k+1}; q^2)_\infty} \times \prod_{k=1}^{n-1} (q^2w^{2k+1}; q^2)_\infty\\
= \quad & \frac{(q^{4n+4})_\infty^{n-1} (q^2w, q^2w^{n+1}, q^2w^{2n+1}; q^2)_\infty}{(q^2)_\infty^{n-2} (q^{4n+4})_\infty} \times \prod_{k=1}^{n-1} (q^2w^{2k+1}; q^2)_\infty\\
= \quad & \frac{(q^{4n+4})_\infty^{n-2} (-q^2; q^2)_\infty}{(q^2)_\infty^{n-2}} \times \prod_{k=0}^{n} (q^2w^{2k+1}; q^2)_\infty. \qquad (3.21)
\end{aligned}
$$

Substituting (3.21) into (3.20) and combining with (3.19), we have a simplified expression for the right hand side of identity (3.15). Comparing with the expression (3.17), we can conclude that the constant

$$c(\tau) = \frac{i^n q^{-\frac{n^2+n}{4}}}{(q^2)_\infty^{n^2+2n}}.$$

□

Theorem 3.1 yielded six classes of identities corresponding to all the infinite families of affine root systems except for A_n. Since we have exhausted all the possibilities for $E^l_{m,j}(z)$ and $O^l_{m,j}(z)$, it is natural to reconsider the basic function $T^l_{m,j}(z)$.

Theorem 3.2 *Let z_j be arbitrary complex variables, then the following identity holds:*

$$
\begin{aligned}
\det_{1\le j,k\le n} \left(T^l_{n,2j-n}(z_k)\right) = \frac{(-i)^{\frac{n^2-n-2}{2}} q^{\frac{-n^2+n-2}{8}}}{(q^2)_\infty^{\frac{n^2-3n+2}{2}}}\, \theta_1\left(\sum_{\ell=1}^n z_\ell \,\middle|\, \tau\right)&\\
\times \prod_{1\le j<k\le n} \theta_1(z_j - z_k|\tau),& \qquad (3.22)
\end{aligned}
$$

where $l \equiv n \pmod 2$.

The infinite product on the right hand side of (3.22) has an extra theta factor when compared to the Macdonald identity for A_{n-1}, nevertheless the two identities are equivalent. Other proofs for the A_{n-1} Macdonald identities can be found in [5], [18] and [21].

Theorems 3.1 and 3.2 fall into the class of elliptic determinant evaluations. A survey of recent progress in this area can be found in [13].

4 Examples

We will conclude by giving several examples of well known identities that can be constructed.

Example 4.1 (Quintuple product identity)

When $m = 3$, $n = 1$, identity (3.2) gives

$$E^1_{3,1}(z) = q^{-\frac{1}{4}}(q^2)^{-2}_\infty \theta_2(z|\tau)\theta_3(z|\tau)\theta_4(z|\tau).$$

This is an alternate form of the quintuple product identity and has appeared in [3], [16] and [20]. To obtain (1.1), replace e^{2iz} by a, q^2 by q, and use the infinite product expansions of $\theta_i(z|\tau)$ listed in (2.2).

Example 4.2 (Septagonal numbers identity)

When $m = 5$, $n = 2$, identity (3.2) gives

$$E^1_{5,1}(x)E^1_{5,3}(y) - E^1_{5,1}(y)E^1_{5,3}(x) = \frac{q^{-1}}{(q^2)^6_\infty}\theta_1(x+y|\tau)\theta_1(x-y|\tau)\theta_2(x|\tau)$$
$$\theta_3(x|\tau)\theta_4(x|\tau)\theta_2(y|\tau)\theta_3(y|\tau)\theta_4(y|\tau).$$

Setting $y = 0$ gives the Hirschhorn-Farkas-Kra septagonal numbers identity (1.2). Farkas and Kra discussed a similar m-th ordered generalization in some detail for even functions, odd m and odd j [9, p.268-271]. In their method, they needed some properties of a specific automorphic form on $\Gamma(m)$. F.G. Garvan [10] has also studied the same problem and using a different method, managed to construct a different class of explicit identities using only the Jacobi triple product identity. However, the identities constructed by both Garvan and Farkas and Kra are identities involving only a single complex variable z.

Example 4.3 (Ewell's identity)

When $m = 4$, $n = 2$, identity (3.3) is

$$E^1_{4,0}(x)E^1_{4,2}(y) - E^1_{4,0}(y)E^1_{4,2}(x) = \frac{2q^{-\frac{1}{2}}(q^4)_\infty}{(q^2)^5_\infty}\theta_1(x+y|\tau)\theta_1(x-y|\tau)$$
$$\theta_3(x|\tau)\theta_4(x|\tau)\theta_3(y|\tau)\theta_4(y|\tau).$$

Setting $y = 0$ gives an identity studied by Ewell [7]. Two different proofs of this identity were given recently in [4].

Example 4.4 (Winquist's identity)

When $m = 3$, $n = 2$, identity (3.5) is

$$O^1_{3,1}(x)O^1_{3,3}(y) - O^1_{3,1}(y)O^1_{3,3}(x) = \frac{-2}{q(q^2)^2_\infty}\theta_1(x+y|\tau)\theta_1(x-y|\tau)\theta_1(x|\tau)\theta_1(y|\tau).$$

This is equivalent to Winquist's identity [24]. Recently, H.H. Chan, S. Cooper and the author has used a variant of this identity to generate interesting representations for $\eta^{10}(\tau)G_k$, the product of the tenth power of the Dedekind's eta-function and some Eisenstein series. (See [2] and the bibliography there for references to Winquist's identity.)

Example 4.5 (Representations of $(q)_\infty^{2n^2-n}$)

One main application of the Macdonald identities was to construct representation of powers of $\eta(\tau)$ or equivalently, powers of $(q)_\infty$. We illustrate this using identity (3.11), the D_n case. For each j, we apply $(\frac{\partial}{\partial z_j})^{2(j-1)}$ to the identity and set $z_j = 0$. This will turn each $\theta_1(z_j \pm z_k|\tau)$ into $\theta_1'(0|\tau)$ which is equal to $2q^{\frac{1}{4}}(q^2)_\infty^3$. Writing q^2 as q, we get the following explicit formula for $(q)_\infty^{2n^2-n}$ when $n \geq 2$,

$$2^{2-n}(q)_\infty^{2n^2-n}\prod_{j=1}^{n-1}(-1)^j(2j)! = \tag{4.1}$$
$$\det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{k-1}\left((2n-2)\ell+j-1\right)^{2k-2}q^{(n-1)\ell^2+(j-1)\ell}\right).$$

A different formula can be obtained from identity (3.2), the BC_n case.

Example 4.6 (Representations of $(q)_\infty^{2n^2+n}$)

For each j, apply $(\frac{\partial}{\partial z_j})^{2j-1}$ to identity (3.5) (the B_n case), and then set $z_j = 0$ to get

$$2^{n^2-n+1}(q)_\infty^{2n^2+n}\prod_{j=1}^{n}(-1)^{j-1}(2j-1)! = \tag{4.2}$$
$$\det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{\ell+k-1}\left((4n-2)\ell+2j-1\right)^{2k-1}q^{((2n-1)\ell^2+(2j-1)\ell)/2}\right).$$

A different formula can be obtained from identity (3.15), the C_n case.

Example 4.7 (Representations of $(q)_\infty^{n^2+2}$)

Due to the extra theta factor in identity (3.22) (our A_{n-1} case), we obtain a representation of $(q)_\infty^{n^2+2}$ instead of $(q)_\infty^{(n-1)^2+2(n-1)}$ given in [17].

$$(-i)^{\frac{n^2-n-2}{2}}2^{\frac{n^2-n+2}{2}}n(q)_\infty^{n^2+2}\prod_{j=1}^{n-1}(-1)^j(j)! = \det_{1\le j,k\le n} f(j,k,n), \tag{4.3}$$

where for $k < n$,

$$
\begin{aligned}
f(j,k,n) &= \sum_{\ell=-\infty}^{\infty} (-1)^{n\ell}\left((2n\ell+2j-n)i\right)^{k-1} q^{(n\ell^2+(2j-n)\ell)/2}, \\
\text{and} \qquad f(j,n,n) &= \sum_{\ell=-\infty}^{\infty} (-1)^{n\ell}\left((2n\ell+2j-n)i\right)^{n} q^{(n\ell^2+(2j-n)\ell)/2}.
\end{aligned}
$$

Besides Macdonald's original paper [17], other representations for powers of $(q)_\infty$ were also given by V.E. Leininger and S.C. Milne in [14] and [15]. In particular, [15] also contains a representation of $(q)_\infty^{n^2+2}$.

Acknowledgement

The author would like to thank Professor H.H. Chan for his advice and invaluable insights and Professor S. Cooper for helpful suggestions and for bringing the work of Rosengren and Schlosser to the author's attention. The author would also like to thank Professor S.-S Huang for providing a copy of his joint work with S.-D. Chen and W.-Y. Chen.

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