RAMANUJAN'S EISENSTEIN SERIES AND POWERS OF DEDEKIND'S ETA-FUNCTION

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Abstract. In this article, we use the theory of elliptic functions to construct theta function identities which are equivalent to Macdonald's identities for $\mathbf{A}_2, \mathbf{B}_2$ and $\mathbf{G}_2$. Using these identities, we express, for $d = 8, 10$ or 14, certain theta functions in the form $\eta^d(\tau)F(P,Q,R)$, where $\eta(\tau)$ is Dedekind's eta-funcion, and $F(P,Q,R)$ is a polynomial in Ramanujan's Eisenstein series P, Q, and R. We also derive identities in the case when $d = 2$. These lead to a new expression for $\eta^{26}(\tau)$. This work generalizes the results for $d = 1$ and $d = 3$ which were given by Ramanujan on page 369 of the "Lost Notebook".

1. Introduction

Let $\mathrm{Im}(\tau) > 0$ and put $q = \exp(2\pi i\tau)$. Dedekind's eta-function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1-q^k),$$

and Ramanujan's Eisenstein series are

$$P = P(q) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k},$$

$$Q = Q(q) = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3q^k}{1-q^k}$$

and

$$R = R(q) = 1 - 504 \sum_{k=1}^{\infty} \frac{k^5q^k}{1-q^k}.$$

On page 369 of The Lost Notebook [28], S. Ramanujan gave the following results:

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Theorem 1.1 (Ramanujan)**.** *Let*

$$S_1(m) = \sum_{\alpha\equiv 1 \pmod 6} (-1)^{(\alpha-1)/6}\alpha^m q^{\alpha^2/24},$$

$$S_3(m) = \sum_{\alpha\equiv 1 \pmod 4} \alpha^m q^{\alpha^2/8}.$$

Then

$$\begin{aligned} S_1(0) &= \eta(\tau), \\ S_1(2) &= \eta(\tau)P, \\ S_1(4) &= \eta(\tau)(3P^2-2Q), \\ S_1(6) &= \eta(\tau)(15P^3-30PQ+16R), \end{aligned}$$

and in general

$$S_1(2m) = \eta(\tau)\sum_{i+2j+3k=m} a_{ijk}P^iQ^jR^k,$$

where a_{ijk} are integers, and i, j and k are non-negative integers. Also,

$$\begin{aligned} S_3(1) &= \eta^3(\tau), \\ S_3(3) &= \eta^3(\tau)P, \\ S_3(5) &= \eta^3(\tau)(5P^2-2Q)/3, \\ S_3(7) &= \eta^3(\tau)(35P^3-42PQ+16R)/9, \end{aligned}$$

and in general

$$S_3(2m+1) = \eta^3(\tau)\sum_{i+2j+3k=m} b_{ijk}P^iQ^jR^k,$$

where b_{ijk} are rational numbers, and i, j and k are non-negative integers.

The results for $S_1(0)$ and $S_3(1)$ are well-known consequences of the Jacobi triple product identity [1, p. 500]. Ramanujan also listed the values of $S_1(8)$, $S_1(10)$, $S_3(9)$ and $S_3(11)$. He indicated that these results may be proved by induction, using differentiation and the Ramanujan differential equations [26, eq. 30]

$$q\frac{dP}{dq} = \frac{P^2-Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ-R}{3}, \qquad q\frac{dR}{dq} = \frac{PR-Q^2}{2}.$$

Theorem 1.1 has been studied by K. Venkatachaliengar [36, pp. 31–32] (where both S_1 and S_3 are studied), B. C. Berndt and A. J. Yee [6] (where S_1 is studied), and B. C. Berndt, S. H. Chan, Z.-G. Liu and H. Yesilyurt [5] (where S_3 is studied). For a different approach to these identities, see Ramanujan [27, Chapter 16, Entry 35 (i)] (for S_3), Berndt [4, p. 61] (for S_3) and Liu [22] (for S_1).

The first purpose of this article is to prove analogous results corresponding to the 2nd, 4th, 6th, 8th, 10th, 14th and 26th powers of $\eta(\tau)$, these being the even powers of $\eta(\tau)$ that are lacunary [33, Theorem 1]. For example, the result for the 14th power is as follows. For non-negative integers m, n, ℓ, let

$$S_{14}(m,n,\ell) = \sum_{\substack{\alpha\equiv 2 \pmod 6\\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6}\left(\beta(\alpha^2-\beta^2)\right)^m\left(\alpha(\alpha^2-9\beta^2)\right)^n \times\left(\alpha^2+3\beta^2\right)^\ell q^{(\alpha^2+3\beta^2)/12}.$$

Then

$$S_{14}(2m+1,2n+1,\ell) = \eta^{14}(\tau)\sum_{i+2j+3k=3m+3n+\ell} c_{ijk}P^iQ^jR^k, \quad m,\, n,\, \ell \geq 0, \tag{1.1}$$

where c_{ijk} are rational numbers, and i, j and k are non-negative integers. The first few instances of (1.1) are:

$$\begin{aligned} S_{14}(1,1,0) &= -30\eta^{14}(\tau), \\ S_{14}(1,1,1) &= -210\eta^{14}(\tau)P, \\ S_{14}(1,1,2) &= -210\eta^{14}(\tau)(8P^2-Q), \\ S_{14}(3,1,0) &= -5\eta^{14}(\tau)(56P^3-21PQ+19R), \\ S_{14}(1,3,0) &= -15\eta^{14}(\tau)(504P^3-189PQ-115R). \end{aligned}$$

An equation equivalent to the one for $S_{14}(1,1,0)$ was stated without proof by L. Winquist [38]. Since

$$\beta(\alpha^2-\beta^2)\alpha(\alpha^2-9\beta^2) = \alpha^5\beta-10\alpha^3\beta^3+9\alpha\beta^5 = \frac{1}{6\sqrt{3}}\mathrm{Im}\left((\alpha+i\beta\sqrt{3})^6\right),$$

the result for $S_{14}(1,1,0)$ may be written as

$$\sum_{\substack{\alpha\equiv 2 \pmod 6\\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6}\,\mathrm{Im}\left((\alpha+i\beta\sqrt{3})^6\right) q^{(\alpha^2+3\beta^2)/12} = -180\sqrt{3}\eta^{14}(\tau).$$

The second purpose of this article is to prove results of the type

$$\begin{aligned} &\sum_{\substack{\alpha\equiv 2 \pmod 6\\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6}\,\mathrm{Im}\left((\alpha+i\beta\sqrt{3})^{6n}\right) q^{(\alpha^2+3\beta^2)/12} \\ &\quad = \sqrt{3}\eta^{14}(\tau)\sum_{2j+3k=3(n-1)} d_{jk}Q^jR^k, \end{aligned} \tag{1.2}$$

where d_{jk} are rational numbers, and j and k are non-negative integers. We shall state analogues of this result for the 2nd, 4th, 6th, 8th, 10th and 26th powers of $\eta(\tau)$, and give a detailed proof for the 10th power.

This work is organized as follows.

Notation and properties of theta functions are established in Section 2.

Sections 3, 4 and 5 are devoted to the 8th, 10th and 14th powers of $\eta(\tau)$, respectively. Each section begins with a multivariate theta function identity which is then used to prove the analogue of (1.1) for the 8th, 10th or 14th power of $\eta(\tau)$.

Section 6 is concerned with the analogues of (1.1) for 2nd, 4th and 6th powers of $\eta(\tau)$. These follow from Ramanujan's Theorem 1.1.

In Section 7 we prove results analogous to (1.2) for the 2nd, 4th, 6th, 8th, 10th and 14th powers of $\eta(\tau)$. Since Ramanujan's Eisenstein series P does not occur in these results, the modular transformation for multiple theta series given by B. Schoeneberg [32] can be used to prove them.

In Section 8 we give a simple proof of a series expansion for $\eta^{26}(\tau)$, as well as analogues of (1.1) and (1.2) for the 26th power of $\eta(\tau)$ which are new. The proofs rely on two different analogues of (1.2) for $\eta^2(\tau)$.

Finally, in Section 9 we make some remarks about lacunary series and the Hecke operator, and a new formula for $\eta^{24}(\tau)$ is presented.

2. Preliminaries

In the classical theory of theta functions [37], the notation $q = \exp(\pi i \tau)$ is used, whereas in the theory of modular forms $q = \exp(2\pi i \tau)$. Because we will use both theories, we let $t = 2\tau$, and define

$$q = \exp(\pi i t) = \exp(2\pi i \tau).$$

We will use t when working with theta functions, and τ for modular forms and Dedekind's η function.

The Jacobi theta functions [1, p. 509], [37, Ch. 21] are defined by

$$\begin{aligned}
\theta_1(z|t) &= 2\sum_{k=0}^{\infty}(-1)^k q^{(k+\frac{1}{2})^2}\sin(2k+1)z,\\
\theta_2(z|t) &= 2\sum_{k=0}^{\infty} q^{(k+\frac{1}{2})^2}\cos(2k+1)z,\\
\theta_3(z|t) &= 1+2\sum_{k=1}^{\infty} q^{k^2}\cos 2kz
\end{aligned}$$

and

$$\theta_4(z|t) = 1+2\sum_{k=1}^{\infty}(-1)^k q^{k^2}\cos 2kz.$$

Let

$$\begin{aligned}G_2(z|t) &= 2\sum_{\alpha\equiv 1 \pmod 6} q^{\alpha^2/12}\sin(\alpha z),\\ G_3(z|t) &= 2\sum_{\alpha\equiv 4 \pmod 6} q^{\alpha^2/12}\sin(\alpha z),\\ H(z|t) &= G_2(4z|4t) - G_3(4z|4t)\\ &= 2\sum_{\alpha\equiv 2 \pmod 6} (-1)^{(\alpha-2)/6} q^{\alpha^2/12}\sin(2\alpha z)\end{aligned}$$

and

$$T(z|t) = \theta_1(2z|t).$$

These functions satisfy the transformation properties

$$\begin{aligned}
&\theta_1(z+\pi|t) = -\theta_1(z|t), && \theta_1(z+\pi t|t) = -q^{-1}e^{-2iz}\theta_1(z|t),\\
&\theta_2(z+\pi|t) = -\theta_2(z|t), && \theta_2(z+\pi t|t) = q^{-1}e^{-2iz}\theta_2(z|t),\\
&\theta_3(z+\pi|t) = \theta_3(z|t), && \theta_3(z+\pi t|t) = q^{-1}e^{-2iz}\theta_3(z|t),\\
&\theta_4(z+\pi|t) = \theta_4(z|t), && \theta_4(z+\pi t|t) = -q^{-1}e^{-2iz}\theta_4(z|t),\\
&G_2(z+\pi|t) = -G_2(z|t), && G_2(z+\pi t|t) = q^{-3}e^{-6iz}G_2(z|t),\\
&G_3(z+\pi|t) = G_3(z|t), && G_3(z+\pi t|t) = q^{-3}e^{-6iz}G_3(z|t),\\
&H\left(z+\frac{\pi}{2}\middle|t\right) = H(z|t), && H\left(z+\frac{\pi t}{2}\middle|t\right) = -q^{-3}e^{-12iz}H(z|t),\\
&T\left(z+\frac{\pi}{2}\middle|t\right) = -T(z|t), && T\left(z+\frac{\pi t}{2}\middle|t\right) = -q^{-1}e^{-4iz}T(z|t).
\end{aligned}$$

By the Jacobi triple product identity [1, p. 497],

$$\theta_1(z|t) = 2q^{1/4}\sin z\prod_{k=1}^{\infty}(1-q^{2k}e^{2iz})(1-q^{2k}e^{-2iz})(1-q^{2k}).$$

Therefore $\theta_1(z|t)$ has simple zeroes at $z=\pi m+\pi tn$, m, $n\in\mathbb{Z}$, and no other zeroes.

We will also need the results

$$\theta_2(z|t)G_2(z|t) = \eta(2\tau)\theta_1(2z|t), \tag{2.1}$$

$$\theta_3(z|t)G_3(z|t) = -\eta(2\tau)\theta_1(2z|t). \tag{2.2}$$

These are equivalent to the quintuple product identity. For example, see [34, Prop. 2.1], where these and two other similar equations are given. Equations (2.1) and (2.2), together with the Jacobi triple product identity, imply $G_2(z|t)$ has simple zeroes when $z=\pi m/2+\pi tn/2$, where m and n are integers and $(m,n)\not\equiv(1,0)\pmod 2$, and no other zeroes. Similarly, $G_3(z|t)$ has simple zeroes when $z=\pi m/2+\pi tn/2$, where m and n are integers and

$(m,n) \not\equiv (1,1)\,(\mathrm{mod}\ 2)$, and no other zeroes. Equations (2.1) and (2.2) also imply

$$\theta_2(z|t)G_2(z|t) + \theta_3(z|t)G_3(z|t) = 0.$$

The following lemma is of fundamental importance and will be used several times in the proofs in the subsequent sections. Let $f^{(\ell)}(z|t)$ denote the ℓ-th derivative of $f(z|t)$ with respect to z.

Lemma 2.1.

$$\begin{aligned} &\theta_1^{(2\ell_1+1)}\left(0\middle|\frac{t}{2}\right)\theta_1^{(2\ell_2+1)}\left(0\middle|\frac{t}{2}\right)\cdots\theta_1^{(2\ell_m+1)}\left(0\middle|\frac{t}{2}\right) \\ = \;& (\eta(\tau))^{3m} \sum_{i+2j+3k=\ell_1+\ell_2+\cdots+\ell_m} a_{ijk}P^iQ^jR^k, \end{aligned}$$

for some rational numbers a_{ijk}, where i, j and k are non-negative integers.

Proof. Let us first consider the case $m = 1$. From the definition of θ_1, we have

$$\theta_1^{(2\ell+1)}(z|t) = 2(-1)^\ell \sum_{k=0}^{\infty}(-1)^k(2k+1)^{2\ell+1}q^{(k+\frac{1}{2})^2}\cos(2k+1)z.$$

Therefore

$$\begin{aligned} \theta_1^{(2\ell+1)}\left(0\middle|\frac{t}{2}\right) &= 2(-1)^\ell \sum_{k=0}^{\infty}(-1)^k(2k+1)^{2\ell+1}q^{(k+\frac{1}{2})^2/2} \\ &= 2(-1)^\ell \sum_{k=-\infty}^{\infty}(4k+1)^{2\ell+1}q^{(4k+1)^2/8} \\ &= 2(-1)^\ell S_3(2\ell+1) \\ &= \eta^3(\tau)\sum_{i+2j+3k=\ell} a_{ijk}P^iQ^jR^k, \end{aligned}$$

by Theorem 1.1. The general case $m \geq 1$ now follows by multiplying m copies of this result together. □

Finally we define the standard notation for products:

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k)$$

and

$$(x_1, x_2, \cdots, x_m; q)_\infty = (x_1;q)_\infty(x_2;q)_\infty\cdots(x_m;q)_\infty.$$

3. The eighth power of $\eta(\tau)$

The main tool used in this section is

Theorem 3.1.

$$G_2(x|t)\theta_2(y|t)+G_3(x|t)\theta_3(y|t)=\frac{1}{\eta(\tau)}\theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right).$$

Proof. Let

$$M_8(x,y|t)=G_2(x|t)\theta_2(y|t)+G_3(x|t)\theta_3(y|t)$$

and

$$N_8(x,y|t)=\theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right).$$

Then the formulas listed in Section 2 imply M_8 and N_8 satisfy the transformation properties

$$f(x+2\pi,y|t)=f(x,y|t),\qquad f(x+\pi t,y|t)=q^{-3}e^{-6ix}f(x,y|t),$$
$$f(x,y+2\pi|t)=f(x,y|t),\qquad f(x,y+\pi t|t)=q^{-1}e^{-2iy}f(x,y|t).$$

Fix y and consider M_8 and N_8 as functions of x. N_8 has simple zeroes at $x=\pi m+\pi tn/2$, $\pm y+2\pi m+\pi tn$, m, $n\in\mathbb{Z}$, and no other zeroes. By the results in Section 2, we see that M_8 also has zeroes at these points, and possibly at other points, too. Therefore $M_8(x,y|t)/N_8(x,y|t)$ is an elliptic function of x with no poles, and thus is a constant independent of x.

Now fix x and consider M_8 and N_8 as functions of y. N_8 has simple zeroes at $y=\pm x+2\pi m+\pi tn$ and no other zeroes. It is easy to check that M_8 also has zeroes at these points, and possibly at other points, too. Therefore M_8/N_8 is an elliptic function of y with no poles and thus is a constant independent of y.

It follows that

$$\frac{M_8(x,y|t)}{N_8(x,y|t)}=C(q)$$

for some $C(q)$ independent of x and y. To calculate $C(q)$, let $x=\pi/2$ and $y=\pi$. Since $G_3(\pi/2|t)=0$, we have

$$\begin{aligned}
M_8\left(\frac{\pi}{2},\pi\middle|t\right) &= G_2\left(\frac{\pi}{2}\middle|t\right)\theta_2(\pi|t)\\
&= -2\sum_{k=-\infty}^{\infty}(-1)^k q^{(6k+1)^2/12}\sum_{j=-\infty}^{\infty}q^{(j+\frac{1}{2})^2}\\
&= -4\eta(2\tau)\,q^{\frac{1}{4}}(-q^2,-q^2,q^2;q^2)_\infty\\
&= -4\eta^2(4\tau).
\end{aligned}$$

On the other hand,

$$\begin{aligned} N_8\left(\frac{\pi}{2},\pi\middle|t\right) &= \theta_1\left(-\frac{\pi}{4}\middle|\frac{t}{2}\right)\theta_1\left(\frac{\pi}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{3\pi}{4}\middle|\frac{t}{2}\right) \\ &= -\left(2q^{\frac{1}{8}}\right)^3 \sin\tfrac{\pi}{4}\sin\tfrac{\pi}{2}\sin\tfrac{3\pi}{4}(iq,-iq,q;q)_\infty^2(-q,-q,q;q)_\infty \\ &= -4\eta(\tau)\eta^2(4\tau), \end{aligned}$$

after simplifying. Therefore

$$C(q) = \frac{M_8(\frac{\pi}{2},\pi|t)}{N_8(\frac{\pi}{2},\pi|t)} = \frac{1}{\eta(\tau)}.$$

This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let m and n be non-negative integers and define*

$$S_8(m,n) = \sum_{\substack{\alpha\equiv 1 \pmod 3\\ \alpha+\beta\equiv 0 \pmod 2}} \alpha^m\beta^n q^{(\alpha^2+3\beta^2)/12}.$$

Then $S_8(1,0)=0$ and

$$S_8(2m+1,2n) = \eta^8(\tau)\sum_{i+2j+3k=m+n-1} a_{ijk}P^iQ^jR^k, \tag{3.1}$$

provided $m+n\ge 1$. Here a_{ijk} are rational numbers, and i,j and k are non-negative integers.

Proof. Apply $\dfrac{\partial^{2m+2n+1}}{\partial x^{2m+1}\partial y^{2n}}$ to the identity in Theorem 3.1 and let $x=y=0$. The left hand side is

$$\begin{aligned} &G_2^{(2m+1)}(0|t)\ \theta_2^{(2n)}(0|\tau) + G_3^{(2m+1)}(0|t)\ \theta_3^{(2n)}(0|\tau) \\ &= 2(-1)^{m+n}\sum_{\alpha\equiv 1 \pmod 6}\alpha^{2m+1}q^{\alpha^2/12}\sum_{\beta\equiv 1 \pmod 2}\beta^{2n}q^{\beta^2/4} \\ &\quad + 2(-1)^{m+n}\sum_{\alpha\equiv 4 \pmod 6}\alpha^{2m+1}q^{\alpha^2/12}\sum_{\beta\equiv 0 \pmod 2}\beta^{2n}q^{\beta^2/4} \\ &= 2(-1)^{m+n}\sum_{\substack{\alpha\equiv 1 \pmod 3\\ \alpha+\beta\equiv 0 \pmod 2}}\alpha^{2m+1}\beta^{2n}q^{(\alpha^2+3\beta^2)/12}. \end{aligned} \tag{3.2}$$

Since $\theta_1(z|t)$ is an odd function, the right hand side is a linear combination of terms of the form

$$\frac{1}{\eta(\tau)}\theta_1^{(2\ell_1+1)}\left(0\middle|\frac{t}{2}\right)\theta_1^{(2\ell_2+1)}\left(0\middle|\frac{t}{2}\right)\theta_1^{(2\ell_3+1)}\left(0\middle|\frac{t}{2}\right)$$

where $(2\ell_1+1)+(2\ell_2+1)+(2\ell_3+1)=2m+2n+1$. By Lemma 2.1, the right hand side is therefore of the form

$$\eta^8(\tau)\sum_{i+2j+3k=m+n-1} a_{ijk}P^iQ^jR^k. \tag{3.3}$$

If we combine (3.2) and (3.3), we complete the proof of the Theorem for the case $m+n \geq 1$. The result for $S_8(1,0)$ is obtained similarly. □

The following identities are consequences of Theorem 3.2.

$$\begin{aligned} S_8(1,0) &= 0, \\ S_8(3,0) &= -6\eta^8(\tau), \\ S_8(5,0) &= -30\eta^8(\tau)P, \\ S_8(7,0) &= -\frac{63}{2}\eta^8(\tau)(5P^2-Q), \\ S_8(7,2) &= 2\eta^8(\tau)R, \\ S_8(5,4) &= \eta^8(\tau)(5P^3-3PQ). \end{aligned}$$

We also have

$$\begin{aligned} & S_8(3,0) : S_8(1,2) = -3 : 1, \\ & S_8(5,0) : S_8(3,2) : S_8(1,4) = -15 : 1 : 1, \\ & S_8(7,0) : S_8(5,2) : S_8(3,4) : S_8(1,6) = -63 : 1 : 1 : 1, \\ & \begin{pmatrix} S_8(9,0) \\ S_8(3,6) \\ S_8(1,8) \end{pmatrix} = \begin{pmatrix} -66 & -189 \\ 1/3 & 2/3 \\ 2/9 & 7/9 \end{pmatrix} \begin{pmatrix} S_8(7,2) \\ S_8(5,4) \end{pmatrix}. \end{aligned}$$

As mentioned in the introduction, an identity equivalent to $S_8(1,2) = 2\eta^8(\tau)$ was stated without proof by Winquist [38]. The formula for $\eta^8(\tau)$ given by F. Klein and R. Fricke [19, p. 373] can be shown to be equivalent to $S_8(3,0) + 27S_8(1,2) = 48\eta^8(\tau)$. Schoeneberg [31, eq. (11)] gave the attractive form

$$\eta^8(\tau) = \frac{1}{6} \sum_{\mu \in \mathbb{Z}[\exp(2\pi i/3)]} \chi(\mu)\mu^3 \exp(2\pi i \tau |\mu|^2/3),$$

where

$$\chi(\mu) = \begin{cases} 1 & \text{if } \mu \equiv 1 \pmod{\sqrt{-3}}, \\ -1 & \text{if } \mu \equiv -1 \pmod{\sqrt{-3}}. \end{cases}$$

(The sum over the terms satisfying $\mu \equiv 0 \,(\mathrm{mod}\ \sqrt{-3})$ is zero.) Schoeneberg's formula can be deduced from the formulas for $S_8(3,0)$ and $S_8(1,2)$.

Theorem 3.1 is equivalent to Macdonald's identity for $\mathbf{A_2}$ (see [10], [11, Theorem 2.1], [23] or [35, p. 146]) in the form

$$\begin{aligned} & (u, qu^{-1}, v, qv^{-1}, uv, qu^{-1}v^{-1}, q, q; q)_\infty \\ & \quad = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{3m^2-3mn+3n^2+m+n} h_{m,n}(u,v), \end{aligned}$$

where $u = e^{i(x+y)}$, $v = e^{i(x-y)}$, and

$$\begin{aligned} h_{m,n}(u,v) = uv\Big\{ &(u^{-3m-1}v^{-3n-1} - u^{3m+1}v^{3n+1}) \\ &+ (u^{3n-3m}v^{3n+1} - u^{3m-3n}v^{-3n-1}) \\ &+ (u^{3n+1}v^{3n-3m} - u^{-3n-1}v^{3m-3n})\Big\}. \end{aligned}$$

4. The tenth Power of $\eta(\tau)$

The main tool used in this section is

Theorem 4.1.

$$\begin{aligned} &G_3(x|t)G_2(y|t) - G_2(x|t)G_3(y|t) \\ &= \frac{1}{\eta^2(\tau)}\theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(y\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right). \end{aligned}$$

Proof. Apply the technique used in the proof of Theorem 3.1. Let

$$M_{10}(x,y|t) := G_3(x|t)G_2(y|t) - G_2(x|t)G_3(y|t)$$

and

$$N_{10}(x,y|t) := \theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(y\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right).$$

Then M_{10} and N_{10} satisfy the transformation formulas:

$$\begin{aligned} f(x+2\pi,y|t) &= f(x,y|t), & f(x+\pi t,y|t) &= q^{-3}e^{-6ix}f(x,y|t), \\ f(x,y+2\pi|t) &= f(x,y|t), & f(x,y+\pi t|t) &= q^{-3}e^{-6iy}f(x,y|t). \end{aligned}$$

Let y be fixed. Then N_{10} has simple zeroes at $x = \pi m + \pi t n/2$, $\pm y + 2\pi m + \pi t n$, $m, n \in \mathbb{Z}$, and no other zeroes. The results in Section 2 imply M_{10} also has zeroes at the same points as N_{10}, and possibly at other points, too. Thus $M_{10}(x,y|t)/N_{10}(x,y|t)$ is an elliptic function of x with no poles, and therefore is a constant which is independent of x.

By the symmetry in x and y, we find that $M_{10}(x,y|t)/N_{10}(x,y|t)$ is also independent of y, and therefore depends only on q. Let us denote the constant by $D(q)$. To determine its value, let $x = \pi/2$ and $y = \pi/6$. Since

$G_3(\pi/2|t) = 0$ we have

$$\begin{aligned} &M_{10}\left(\frac{\pi}{2}, \frac{\pi}{6}\middle| t\right) \\ &= -G_2\left(\frac{\pi}{2}\middle| t\right) G_3\left(\frac{\pi}{6}\middle| t\right) \\ &= -4 \sum_{j=-\infty}^{\infty} q^{(6j+1)^2/12} \sin(3j+\tfrac{1}{2})\pi \sum_{k=-\infty}^{\infty} q^{(6k-2)^2/12} \sin(k-\tfrac{1}{3})\pi \\ &= 2\sqrt{3}\left(q^{\frac{1}{12}} \sum_{j=-\infty}^{\infty} (-1)^j q^{3j^2+j}\right)\left(q^{\frac{1}{3}} \sum_{k=-\infty}^{\infty} (-1)^k q^{3k^2-2k}\right) \\ &= 2\sqrt{3}\,\eta(2\tau) q^{\frac{1}{3}} (q, q^5, q^6; q^6)_\infty \\ &= 2\sqrt{3}\,\frac{\eta(\tau)\eta^2(6\tau)}{\eta(3\tau)}. \end{aligned}$$

On the other hand, writing $\gamma = \exp(i\pi/3)$ we have

$$\begin{aligned} &N_{10}\left(\frac{\pi}{2}, \frac{\pi}{6}\middle| t\right) \\ &= \theta_1\left(\frac{\pi}{6}\middle|\frac{t}{2}\right)^2 \theta_1\left(\frac{\pi}{3}\middle|\frac{t}{2}\right)\theta_1\left(\frac{\pi}{2}\middle|\frac{t}{2}\right) \\ &= \left(2q^{\frac{1}{8}}\right)^4 \sin^2\tfrac{\pi}{6}\sin\tfrac{\pi}{3}\sin\tfrac{\pi}{2}(\gamma q, \gamma^5 q, q; q)_\infty^2 (\gamma^2 q, \gamma^4 q, q; q)_\infty (\gamma^3 q, \gamma^3 q, q; q)_\infty \\ &= 2\sqrt{3}\,\frac{\eta^3(\tau)\eta^2(6\tau)}{\eta(3\tau)}, \end{aligned}$$

after simplifying the infinite products. So

$$D(q) = \frac{M_{10}(\frac{\pi}{3}, \frac{\pi}{6}|t)}{N_{10}(\frac{\pi}{3}, \frac{\pi}{6}|t)} = \frac{1}{\eta^2(\tau)}.$$

□

Theorem 4.2. *Let*

$$S_{10}(m,n) = \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 4 \pmod 6}} (\alpha^m\beta^n - \alpha^n\beta^m) q^{(\alpha^2+\beta^2)/12}.$$

Then

$$S_{10}(2m+1, 2n+1) = \eta^{10}(\tau) \sum_{i+2j+3k=m+n-1} a_{ijk} P^i Q^j R^k, \tag{4.1}$$

where a_{ijk} are rational numbers, and i, j and k are non-negative integers.

Proof. Apply $\dfrac{\partial^{2m+2n+2}}{\partial x^{2m+1}\partial y^{2n+1}}$ to both sides of Theorem 4.1, then let $x = y = 0$. We omit the details as they are similar to those in the proof of Theorem 3.2. □

The first few examples of Theorem 4.2 are:

$$\begin{aligned}
S_{10}(3,1) &= 6\eta^{10}(\tau),\\
S_{10}(5,1) &= 30\eta^{10}(\tau)P,\\
S_{10}(7,1) &= \frac{63}{2}\eta^{10}(\tau)(5P^2-Q),\\
S_{10}(5,3) &= \frac{3}{2}\eta^{10}(\tau)(15P^2+Q),\\
S_{10}(9,1) &= 3\eta^{10}(\tau)(315P^3-189PQ+44R),\\
S_{10}(7,3) &= \frac{3}{2}\eta^{10}(\tau)(105P^3-21PQ-4R).
\end{aligned}$$

Theorem 4.1 is equivalent to Winquist's identity [38, Theorem 1.1]: put $a = e^{i(x+y)}$, $b = e^{i(x-y)}$ in Theorem 4.1 to get [38, Theorem 1.1]. Observe that the left hand side of Theorem 4.1 is a difference of two terms, and each term is a product of two series that can be summed by the quintuple product identity. This was first noticed by S.-Y. Kang [18]. More information on Winquist's identity can be found in [5], [7], [9], [14], [17], [20] and [21].

5. The fourteenth Power of $\eta(\tau)$

The main tool used in this section is

Theorem 5.1.

$$\begin{aligned}
&H(x|t)T(y|t) + H\left(\frac{x-y}{2}\middle|t\right)T\left(\frac{3x+y}{2}\middle|t\right) + H\left(\frac{x+y}{2}\middle|t\right)T\left(\frac{-3x+y}{2}\middle|t\right)\\
&= \frac{1}{\eta^4(\tau)}\theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(y\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{3x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{-3x+y}{2}\middle|\frac{t}{2}\right).
\end{aligned}$$

Proof. Apply the elliptic function method used in the previous two sections. By the results in Section 2, it may be checked that both sides satisfy the transformation formulas

$$\begin{aligned}
f(x+2\pi,y|t) &= f(x,y|t), & f(x+\pi t,y|t) &= q^{-12}e^{-24ix}f(x,y|t),\\
f(x,y+2\pi|t) &= f(x,y|t), & f(x,y+\pi t|t) &= q^{-4}e^{-8iy}f(x,y|t).
\end{aligned}$$

It is straightforward to check that for a fixed value of x or y, the left hand side is zero whenever the right hand side is zero. Finally, the constant may be evaluated by letting $x = -\pi/8$, $y = 7\pi/8$. □

Because the left hand side of Theorem 5.1 is more complicated than the left hand sides of Theorems 3.1 and 4.1, some extra analysis is needed before differentiating. We will need:

Lemma 5.2. *Let $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$. Let $f(z)$ and $g(z)$ be analytic functions. Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \right\}.$$

Then

$$\begin{aligned} (5.1) \qquad & D_x D_y (D_x^2 - D_y^2)(D_x^2 - 9D_y^2)\left(f(ax+by)g(cx+dy)\right) \\ &= f^{(5)}(ax+by)g'(cx+dy) - 10f'''(ax+by)g'''(cx+dy) \\ &\quad +9f'(ax+by)g^{(5)}(cx+dy). \end{aligned}$$

More generally, for non-negative integers m, n and ℓ, define an operator $D_{x,y}(m,n,\ell)$ and coefficients $c_{i,j}(m,n,\ell)$ by

$$\begin{aligned} D_{x,y}(m,n,\ell) &= \left(D_y(D_x^2 - D_y^2)\right)^m \left(D_x(D_x^2 - 9D_y^2)\right)^n \left(D_x^2 + 3D_y^2\right)^\ell \\ &= \sum_{i+j=3m+3n+2\ell} c_{i,j}(m,n,\ell) D_x^i D_y^j. \end{aligned}$$

Then

(5.2)

$$\begin{aligned} & D_{x,y}(2m+1, 2n+1, \ell)\left(f(ax+by)g(cx+dy)\right) \\ &= \sum_{i+j=6(m+n+1)+2\ell} c_{i,j}(2m+1, 2n+1, \ell)\left(f^{(i)}(ax+by)g^{(j)}(cx+dy)\right). \end{aligned}$$

Proof. The result is trivial if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In either of the other cases, calculations using the chain rule imply that

$$\begin{aligned} (5.3) \qquad & D_y(D_x^2 - D_y^2)\left(f(ax+by)g(cx+dy)\right) \\ &= -f''(ax+by)g'(cx+dy) + f(ax+by)g'''(cx+dy), \\ (5.4) \qquad & D_x(D_x^2 - 9D_y^2)\left(f(ax+by)g(cx+dy)\right) \\ &= -f'''(ax+by)g(cx+dy) + 9f'(ax+by)g''(cx+dy), \\ (5.5) \qquad & (D_x^2 + 3D_y^2)\left(f(ax+by)g(cx+dy)\right) \\ &= f''(ax+by)g(cx+dy) + 3f(ax+by)g''(cx+dy). \end{aligned}$$

If we combine (5.3) and (5.4), we obtain (5.1), which is the case $m = n = \ell = 0$ of (5.2). The general result (5.2) now follows by induction on m, n and ℓ, using (5.3)–(5.5). □

Theorem 5.3. *Let*

$$S_{14}(m,n,\ell) = \sum_{\substack{\alpha \equiv 2 \pmod 6 \\ \beta \equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6} \left(\beta(\alpha^2 - \beta^2)\right)^m \left(\alpha(\alpha^2 - 9\beta^2)\right)^n \left(\alpha^2 + 3\beta^2\right)^\ell q^{(\alpha^2+3\beta^2)/12}.$$

Then

$$S_{14}(2m+1,2n+1,\ell) = \eta^{14}(\tau) \sum_{i+2j+3k=3m+3n+\ell} a_{ijk} P^i Q^j R^k, \tag{5.6}$$

where a_{ijk} are rational numbers, and i, j and k are non-negative integers.

Proof. Apply the operator $D_{x,y}(2m+1, 2n+1, \ell)$ to the identity in Theorem 5.1, then let $x = y = 0$. For the left hand side, use Lemma 5.2 and for the right hand side use Lemma 2.1. □

Since

$$(\alpha^2+3\beta^2)^3 = 27\beta^2(\alpha^2-\beta^2)^2 + \alpha^2(\alpha^2-9\beta^2)^2,$$

it follows that

$$S_{14}(2m+1,2n+1,\ell+3) = 27S_{14}(2m+3,2n+1,\ell) + S_{14}(2m+1,2n+3,\ell).$$

Therefore without loss of generality we may assume $0 \le \ell \le 2$.

The first few examples of Theorem 5.3 were given in Section 1. Theorem 5.1 is equivalent to Macdonald's identity for $\mathbf{G_2}$ (see [11, (1.8)]) written in the form

$$\begin{aligned}(u, qu^{-1}, uv, &qu^{-1}v^{-1}, u^2v, qu^{-2}v^{-1}, u^3v, qu^{-3}v^{-1}, v, qv^{-1}, u^3v^2, qu^{-3}v^{-2}, q, q; q)_\infty \\ &= \sum_m \sum_n q^{12m^2-12mn+4n^2-m-n} H_{m,n}(u,v),\end{aligned}$$

where $u = e^{2ix}$, $v = e^{i(y-3x)}$ and

$$\begin{aligned}H_{m,n}(u,v) = u^5v^3\Big\{ &(u^{12m-5}v^{4n-3} + u^{-12m+5}v^{-4n+3}) \\ &- (u^{12n-12m-4}v^{4n-3} + u^{12m-12n+4}v^{-4n+3}) \\ &+ (u^{12n-12m-4}v^{8n-12m-1} + u^{12m-12n+4}v^{12m-8n+1}) \\ &- (u^{12n-24m+1}v^{8n-12m-1} + u^{24m-12n-1}v^{12m-8n+1}) \\ &+ (u^{12n-24m+1}v^{4n-12m+2} + u^{24m-12n-1}v^{12m-4n-2}) \\ &- (u^{-12m+5}v^{4n-12m+2} + u^{12m-5}y^{12m-4n-2})\Big\}.\end{aligned}$$

6. Second, fourth and sixth powers of $\eta(\tau)$

Analogous results for $\eta^2(\tau)$, $\eta^4(\tau)$ and $\eta^6(\tau)$ can be obtained trivially by multiplying Ramanujan's results for S_1 and S_3. Specifically, let

$$\begin{aligned}S_2(m,n) &= S_1(m)S_1(n), \\ S_4(m,n) &= S_1(m)S_3(n), \\ S_6(m,n) &= S_3(m)S_3(n).\end{aligned}$$

Then

$$S_2(2m,2n) = \eta^2(\tau) \sum_{i+2j+3k=m+n} a_{ijk} P^i Q^j R^k, \tag{6.1}$$

$$S_4(2m,2n+1) = \eta^4(\tau) \sum_{i+2j+3k=m+n} a_{ijk} P^i Q^j R^k, \tag{6.2}$$

$$S_6(2m+1,2n+1) = \eta^6(\tau) \sum_{i+2j+3k=m+n} a_{ijk} P^i Q^j R^k. \tag{6.3}$$

In each case, a_{ijk} are rational numbers, and i, j and k are non-negative integers.

Another form for $\eta^6(\tau)$ was given by Schoeneberg [31, eq. (8)]:

$$\eta^6(\tau) = \frac{1}{2} \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \mathrm{Re}(a+2ib)^2 q^{(a^2+4b^2)/4}.$$

This formula can be shown to be equivalent to the identity for $S_6(1,1)$ by direct series manipulations.

Results of a different type for $\eta^6(\tau)$ may be obtained using a series given by M. Hirschhorn [16]. Let

$$S_6^*(m,n) = \sum_{\substack{\alpha\equiv 1 \pmod{10} \\ \beta\equiv 3 \pmod{10}}} (-1)^{(\alpha+\beta-4)/10} (\alpha^m \beta^n - \alpha^n \beta^m) q^{(\alpha^2+\beta^2)/40}.$$

Hirschhorn's result is

$$S_6^*(0,2) = 8\eta^6(\tau).$$

Using the techniques in this paper it can be shown that if $m+n \geq 1$, then

$$S_6^*(2m,2n) = \eta^6(\tau) \sum_{i+2j+3k=m+n-1} a_{ijk} P^i Q^j R^k,$$

where a_{ijk} are rational numbers, and i, j and k are non-negative integers.

7. Identities obtained using Schoeneberg's theta functions

In this section we prove (1.2) and analogous results for 2nd, 4th, 6th, 8th and 10th powers of $\eta(\tau)$. Most of the results in this section are new. A few special cases can be found in Ramanujan's Lost Notebook, for example [28, p. 249]. Some of Ramanujan's identities have recently been examined by S. S. Rangachari [29], [30], using Hecke's theta functions [15].

The results we shall prove are as follows.

Theorem 7.1. *Let*

$$C_2(n|\tau) = \sum_{\substack{\alpha\equiv 1 \pmod{6} \\ \beta\equiv 1 \pmod{6}}} (-1)^{(\alpha+\beta-2)/6} (\alpha+i\beta)^n q^{(\alpha^2+\beta^2)/24}.$$

Then $C_2(4n|\tau)/\eta^2(\tau)$ is a modular form of weight $4n$ on $SL_2(\mathbb{Z})$.

Theorem 7.2. *Let*

$$C_2^*(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 0 \pmod 6\\ \beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-1)/6}(\alpha+i\beta\sqrt{3})^n q^{(\alpha^2+3\beta^2)/36}.$$

Then $C_2^*(6n|\tau)/\eta^2(\tau)$ *is a modular form of weight* $6n$ *on* $SL_2(\mathbb{Z})$.

Theorem 7.3. *Let*

$$C_4(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-1)/6}\mathrm{Im}\left((\alpha+i\beta\sqrt{3})^n\right) q^{(\alpha^2+3\beta^2)/24}.$$

Then $C_4(2n+1|\tau)/\eta^4(\tau)$ *is a modular form of weight* $2n$ *on* $SL_2(\mathbb{Z})$.

Theorem 7.4. *Let*

$$C_6(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 1 \pmod 4\\ \beta\equiv 1 \pmod 4}} (\alpha+i\beta)^n q^{(\alpha^2+\beta^2)/8}.$$

Then $C_6(4n+2|\tau)/\eta^6(\tau)$ *is a modular form of weight* $4n$ *on* $SL_2(\mathbb{Z})$.

Theorem 7.5. *Let*

$$C_8(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 1 \pmod 3\\ \alpha+\beta\equiv 0 \pmod 2}} (\alpha+i\beta\sqrt{3})^n q^{(\alpha^2+3\beta^2)/12}.$$

Then $C_8(6n+3|\tau)/\eta^8(\tau)$ *is a modular form of weight* $6n$ *on* $SL_2(\mathbb{Z})$.

Theorem 7.6. *Let*

$$C_{10}(n|\tau) = \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 4 \pmod 6}} \mathrm{Im}\left((\alpha+i\beta)^n\right) q^{(\alpha^2+\beta^2)/12}.$$

Then $C_{10}(4n+4|\tau)/\eta^{10}(\tau)$ *is a modular form of weight* $4n$ *on* $SL_2(\mathbb{Z})$.

Theorem 7.7. *Let*

$$C_{14}(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 2 \pmod 6\\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6}\,\mathrm{Im}\left((\alpha+i\beta\sqrt{3})^n\right) q^{(\alpha^2+3\beta^2)/12}.$$

Then $C_{14}(6n+6|\tau)/\eta^{14}(\tau)$ *is a modular form of weight* $6n$ *on* $SL_2(\mathbb{Z})$.

In order to prove Theorems 7.1–7.7, we first recall some properties of a class of theta functions studied by B. Schoeneberg [32].

Let f be an even positive integer and $A = (a_{\mu,\nu})$ be a *symmetric* $f \times f$ matrix such that

1. $a_{\mu,\nu} \in \mathbb{Z}$;
2. $a_{\mu,\mu}$ is even; and
3. $\mathbf{x}^t A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^f$ such that $\mathbf{x} \neq \mathbf{0}$.

Let N be the smallest positive integer such that NA^{-1} also satisfies conditions 1—3. Let

$$P_k^A(\mathbf{x}) := \sum_{\mathbf{y}} c_{\mathbf{y}}(\mathbf{y}^t A\mathbf{x})^k,$$

where the sum is over finitely many $\mathbf{y} \in \mathbb{C}^f$ with the property $\mathbf{y}^t A\mathbf{y} = 0$, and $c_{\mathbf{y}}$ are arbitrary complex numbers.

When $A\mathbf{h} \equiv \mathbf{0} \pmod N$ and $\operatorname{Im}\tau > 0$, we define

$$\vartheta_{A,\mathbf{h},P_k^A}(\tau) = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^f \\ \mathbf{n}\equiv\mathbf{h} \pmod N}} P_k^A(\mathbf{n}) e^{\frac{2\pi i\tau}{N}\frac{1}{2}\frac{\mathbf{n}^t A\mathbf{n}}{N}}.$$

The result which we need is the following [32, p. 210, Theorem 2]:

Theorem 7.8. *The function $\vartheta_{A,\mathbf{h},P_k^A}$ satisfies the following transformation formulas:*

$$\vartheta_{A,\mathbf{h},P_k^A}(\tau+1) = e^{\frac{2\pi i}{N}\frac{1}{2}\frac{\mathbf{h}^t A\mathbf{h}}{N}}\vartheta_{A,\mathbf{h},P_k^A}(\tau)$$

and

$$\vartheta_{A,\mathbf{h},P_k^A}\left(-\frac{1}{\tau}\right) = \frac{(-i)^{\frac{f}{2}+2k}\tau^{\frac{f}{2}+k}}{\sqrt{|\det A|}} \sum_{\substack{\mathbf{g} \pmod N \\ A\mathbf{g}\equiv\mathbf{0} \pmod N}} e^{\frac{2\pi i}{N}\frac{\mathbf{g}^t A\mathbf{h}}{N}}\vartheta_{A,\mathbf{g},P_k^A}(\tau).$$

We will also need:

Lemma 7.9. *Let*

$$\varphi_{r,s}(n;\tau) = \sum_{\substack{\alpha\equiv r \pmod{12} \\ \beta\equiv s \pmod{12}}} (\alpha - i\beta)^n e^{\frac{2\pi i\tau}{12}\frac{1}{2}\frac{6(\alpha^2+\beta^2)}{12}}.$$

Then

(7.1)

$$\varphi_{r,s}(4n;\tau+1) = e^{6\pi i(r^2+s^2)/12^2}\varphi_{r,s}(4n;\tau)$$

and

(7.2)

$$\varphi_{r,s}\left(4n;-\frac{1}{\tau}\right) = \frac{(-i)\tau^{4n+1}}{6} \sum_{\substack{(u,v) \pmod{12} \\ (6u,6v)\equiv(0,0) \pmod{12}}} e^{\pi i(ru+sv)/12}\varphi_{u,v}(4n;\tau).$$

Proof. These follow from Theorem 7.8 on taking

$$A = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} r \\ s \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} i \\ 1 \end{pmatrix},$$

$N = 12$, $k = 4n$, and $f = 2$. □

We are now ready to prove Theorems 7.1–7.7. We shall give a detailed proof of Theorem 7.6. The details for the other theorems are similar.

Proof of Theorem 7.6. From the first example following Theorem 4.2 and the definition of $C_{10}(4|\tau)$, it follows that

$$C_{10}(4|\tau) = 24\eta^{10}(\tau). \tag{7.3}$$

Next, observe that

(7.4)

$$\begin{aligned} &C_{10}(4n|\tau)\\ &= \frac{1}{2i}\left(\sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 4 \pmod 6}} (\alpha+i\beta)^{4n} q^{(\alpha^2+\beta^2)/12} - \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 4 \pmod 6}} (\alpha-i\beta)^{4n} q^{(\alpha^2+\beta^2)/12}\right)\\ &= \frac{1}{2^{4n+1}i}\left(\sum_{\substack{\alpha\equiv 8 \pmod{12}\\ \beta\equiv 2 \pmod{12}}} (\alpha-i\beta)^{4n} q^{3(\alpha^2+\beta^2)/12^2} - \sum_{\substack{\alpha\equiv 2 \pmod{12}\\ \beta\equiv 8 \pmod{12}}} (\alpha-i\beta)^{4n} q^{3(\alpha^2+\beta^2)/12^2}\right)\\ &= \frac{1}{2^{4n+1}i}\left(\varphi_{8,2}(4n;\tau) - \varphi_{2,8}(4n;\tau)\right). \end{aligned}$$

Equation (7.1) implies

$$\varphi_{8,2}(4n;\tau+1) - \varphi_{8,2}(4n;\tau+1) = e^{5\pi i/6}\left(\varphi_{8,2}(4n;\tau) - \varphi_{2,8}(4n;\tau)\right). \tag{7.5}$$

Equation (7.2) gives

$$\begin{aligned} &\varphi_{8,2}\left(4n;-\frac{1}{\tau}\right) - \varphi_{2,8}\left(4n;-\frac{1}{\tau}\right)\\ &= -\frac{i\tau^{4n+1}}{6}\sum_{j=1}^{6}\sum_{k=1}^{6}\left(e^{\pi i(4j+k)/3} - e^{\pi i(j+4k)/3}\right)\varphi_{2j,2k}(4n;\tau). \end{aligned}$$

If we use the relation $\varphi_{r,s}(4n;\tau) = \varphi_{12-r,12-s}(4n;\tau)$ and simplify, we find that

$$\begin{aligned} &\varphi_{8,2}\left(4n;-\frac{1}{\tau}\right) - \varphi_{2,8}\left(4n;-\frac{1}{\tau}\right)\\ &= -\frac{i\tau^{4n+1}}{6}\Big(4(\varphi_{2,4}-\varphi_{4,2})(4n;\tau) + 2(\varphi_{8,2}-\varphi_{2,8})(4n;\tau)\\ &\qquad +2(\varphi_{12,2}-\varphi_{2,12})(4n;\tau) + 2(\varphi_{4,6}-\varphi_{6,4})(4n;\tau)\\ &\qquad +2(\varphi_{6,12}-\varphi_{12,6})(4n;\tau)\Big). \end{aligned}$$

It is easy to check that

$$\begin{aligned}\varphi_{2,12}(4n;\tau) &= \varphi_{12,2}(4n;\tau),\\ \varphi_{4,6}(4n;\tau) &= \varphi_{6,4}(4n;\tau),\\ \varphi_{6,12}(4n;\tau) &= \varphi_{12,6}(4n;\tau),\\ \varphi_{2,4}(4n;\tau) &= \varphi_{8,2}(4n;\tau),\\ \varphi_{4,2}(4n;\tau) &= \varphi_{2,8}(4n;\tau).\end{aligned}$$

Therefore
(7.6)

$$\varphi_{8,2}\left(4n;-\frac{1}{\tau}\right)-\varphi_{2,8}\left(4n;-\frac{1}{\tau}\right)=-i\tau^{4n+1}\left(\varphi_{8,2}(4n;\tau)-\varphi_{2,8}(4n;\tau)\right).$$

Equations (7.3), (7.4), (7.5) and (7.6) imply that the function

$$F(\tau):=\frac{C_{10}(4n|\tau)}{\eta^{10}(\tau)}$$

satisfies the transformation properties

$$F(\tau+1)=F(\tau),\quad F\left(-\frac{1}{\tau}\right)=\tau^{4n-4}F(\tau).$$

That is, $F(\tau)$ is a modular form of weight $4n-4$ on $\mathrm{SL}_2(\mathbb{Z})$. This completes the proof of Theorem 7.6. ☐

8. The twenty-sixth Power of $\eta(\tau)$

The analogue of (1.2) for the 26th power of $\eta(\tau)$ is:

Theorem 8.1. *For $n \geq 1$, the function*

$$\frac{1}{\eta^{26}(\tau)}\left(\frac{C_2^*(12n|\tau)}{3^{6n}}-(-1)^n\frac{C_2(12n|\tau)}{2^{6n}}\right)$$

is a modular form of weight $12n-12$ on $SL_2(\mathbb{Z})$.

Proof. Calculations using Theorems 7.1 and 7.2 imply that the first few terms in the q-expansions are

$$\begin{aligned}C_2(12n|\tau) &= (-64)^n q^{1/12}\left(1-\left((2+3i)^{12n}+(2-3i)^{12n}\right)q\right.\\ &\qquad\left.+\left(5^{12n}-(4+3i)^{12n}-(4-3i)^{12n}\right)q^2+\cdots\right),\\ C_2^*(12n|\tau) &= (729)^n q^{1/12}\left(1-\left((1+2i\sqrt{3})^{12n}+(1-2i\sqrt{3})^{12n}\right)q\right.\\ &\qquad\left.-5^{12n}q^2+\cdots\right).\end{aligned}$$

The q^2 terms in the two expansions are different because $((4+3i)/5)^{12n}\neq 1$ for any integer n [25, Corollary 3.12]. Therefore $C_2(12n|\tau)$ and $C_2^*(12n|\tau)$ are linearly independent. It follows that

$$\frac{1}{\eta^{2}(\tau)}\left(\frac{C_2^*(12n|\tau)}{3^{6n}}-(-1)^n\frac{C_2(12n|\tau)}{2^{6n}}\right)$$

is a cusp form of weight $12n$ on $\mathrm{SL}_2(\mathbb{Z})$, and so must be of the form $\eta^{24}(\tau)F$, where F is a modular form of weight $12n-12$. This completes the proof. $\square$

Corollary 8.2.

$$\eta^{26}(\tau) = \frac{1}{16308864}\left(\frac{C_2(12|\tau)}{64} + \frac{C_2^*(12|\tau)}{729}\right).$$

Proof. Take $n = 1$ in Theorem 8.1 and observe that

$$(2+3i)^{12} + (2-3i)^{12} - (1+2i\sqrt{3})^{12} - (1-2i\sqrt{3})^{12} = 16308864.$$

$\square$

Corollary 8.2 was discovered and proved in [8]. An equivalent form of this identity had been discovered in 1966 by Atkin [2] (unpublished), and the first published proof was given in 1985 by J.-P. Serre [33]. The proof we have given here is different from those in the literature.

Here is the analogue of (1.1) for $\eta^{26}(\tau)$.

Corollary 8.3. *Let n and ℓ be integers satisfying $n \geq 1$, $\ell \geq 0$, and define*

$$S_{26}(n,\ell) = \left(q\frac{d}{dq}\right)^{\ell}\left(\frac{C_2^*(12n|\tau)}{3^{6n}} - (-1)^n\frac{C_2(12n|\tau)}{2^{6n}}\right).$$

Then

$$S_{26}(n,\ell) = \eta^{26}(\tau)\sum_{i+2j+3k=6(n-1)+\ell} a_{ijk}P^iQ^jR^k, \tag{8.1}$$

where a_{ijk} are rational numbers, and i, j and k are non-negative integers.

Proof. This follows immediately from Theorem 8.1 and the Ramanujan differential equations. $\square$

9. Concluding remarks

9.1. **Lacunarity and the Hecke operator.** By a theorem of Landau [3, p. 244, Theorem 10.5], all of the series $S_2(2m,2n)$, $S_4(2m,2n+1)$, $S_6(2m+1,2n+1)$, $S_8(2m+1,2n)$, $S_{10}(2m+1,2n+1)$, $S_{14}(2m+1,2n+1,\ell)$ and $S_{26}(n,\ell)$ are lacunary. Hence the corresponding expressions on the right hand sides of (3.1), (4.1), (5.6), (6.1)–(6.3) and (8.1) are lacunary.

Let us write

$$S_{14}(2m+1,2n+1,\ell) = Aq^{7/12}\sum_{k=0}^{\infty} a(k)q^k,$$

where A is a numerical constant selected to make $a(0) = 1$. Then the technique used in [12] implies that if $p \equiv 5 \pmod 6$ is prime, then

$$a\left(pk + \frac{7}{12}(p^2-1)\right) = (-1)^{(p+1)/6}p^{6(m+n+1)+2\ell}a\left(\frac{k}{p}\right).$$

Similar results for S_2, S_4, S_6, S_8, S_{10} and S_{26} may also be written down. These results generalize a theorem of M. Newman [24].

9.2. **Ramanujan's τ function.** Ramanujan's function $\tau(n)$ is defined by

$$q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n.$$

If we multiply the results for $S_{10}(3,1)$ and $S_{14}(1,1,0)$, we obtain

$$\eta^{24}(\tau)=-\frac{1}{180}\sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 4 \pmod 6\\ \gamma\equiv 2 \pmod 6\\ \delta\equiv 1 \pmod 4}}(-1)^{(\gamma-2)/6}\alpha\beta(\alpha^2-\beta^2)\gamma\delta(\gamma^2-\delta^2)(\gamma^2-9\delta^2)$$
$$\times q^{(\alpha^2+\beta^2+\gamma^2+3\delta^2)/12}.$$

If we extract the coefficient of q^n on both sides we obtain

$$\tau(n)=-\frac{1}{4320\sqrt{3}}\sum(-1)^{(\gamma-2)/6}\mathrm{Im}\left((\alpha+i\beta)^4\right)\mathrm{Im}\left((\gamma+i\delta\sqrt{3})^6\right)$$

where the summation is over integers satisfying

$$\alpha^2+\beta^2+\gamma^2+3\delta^2=12n,$$

$$\alpha\equiv 1 \pmod 6,\ \beta\equiv 4 \pmod 6,\ \gamma\equiv 2 \pmod 6,\ \delta\equiv 1 \pmod 4.$$

This is different from the representation given by F. J. Dyson [13, p. 636].

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