π is Irrational

A nice proof of the irrationality of π due to Ivan Niven.

Suppose $\pi = \frac{a}{b}$, we define a polynomial $f(x) = \frac{x^n(a-bx)^n}{n!}$.
Let

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

Here we are taking even derivatives. It is clear that each $f^{(j)}(0)$ is an integer. Moreover, since $f(x) = f(\frac{a}{b} - x)$ we have $f^{(j)}(\frac{a}{b}) \in \mathbb{Z}$.

$$\begin{aligned} \frac{d}{dx}\left(F'(x)\sin x - F(x)\cos x\right) &= F''(x)\sin x + F(x)\sin x \\ &= f(x)\sin x \\ \therefore \int_0^\pi f(x)\sin x &= F(\pi) + F(0) = F(\frac{a}{b}) + F(0) \in \mathbb{Z} \end{aligned}$$

But for $0 < x < \pi$, we have $0 < f(x)\sin x < \frac{\pi^n a^n}{n!}$. Thus for large n, $f(x)\sin x$ is arbitrarily small but positive, which contradicts that fact that its integral is an integer.

March 15, 2004